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Yang–Mills solutions on Minkowski space via non-compact coset spaces (2206.12009)

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Introduction

- There are but a few analytic solutions of vacuum Yang–Mills equation in Minkowski space, e.g. with $\mathrm{SU}(2)$ gauge group[1].
- We improve this understanding here by working with non-compact gauge group $\mathrm{SO}(1,3)$, i.e. the Lorentz group.
- The Lorentz group $\mathrm{SO}(1,3)$ is relevant for the gauge-theory formulation of GR. This could also be important in some theories of emergent/modified theories of gravity including supergravity and matrix models.
- These solutions are constructed algebraically but they belong to geometrically distinguished classes.

[1] Tatiana A. Ivanova, Olaf Lechtenfeld and Alexander D. Popov, *Phys. Rev. Lett.* **119** (2017) 061601.

Lightcone interior foliated with $H^3 \cong \mathrm{SO}(1,3)/\mathrm{SO}(3)$

The interior of the lightcone $\mathcal{T}$ can be foliated with unit-hyperboloids H^3,

$$y \cdot y \equiv \eta_{\mu\nu} y^\mu y^\nu = -1, \;\; \mu, \nu = 0, 1, 2, 3 \tag{1}$$

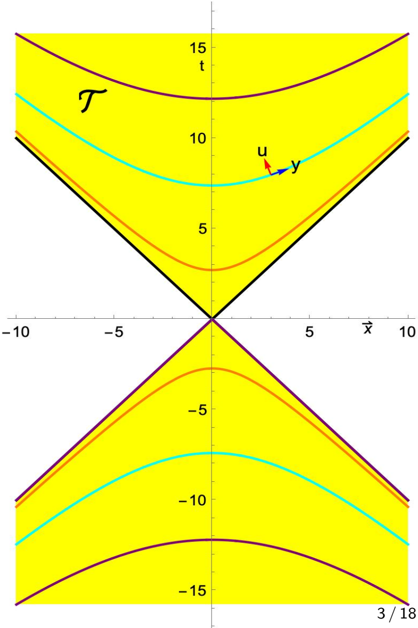
where $\eta = (-,+,+,+)$, using the map

$$\begin{aligned} \varphi_{\mathcal{T}} : \;\; \mathbb{R} \times H^3 &\to \mathcal{T} \;, \\ (u, y^\mu) &\mapsto \mathrm{e}^u \, y^\mu =: x^\mu \end{aligned} \tag{2}$$

and its inverse

$$\begin{aligned} \varphi_{\mathcal{T}}^{-1} : \;\; \mathcal{T} &\to \mathbb{R} \times H^3 \;, \\ x^\mu &\mapsto \left(\ln |x|, \frac{x^\mu}{|x|} \right) \;, \end{aligned} \tag{3}$$

where $|x| := \sqrt{|x \cdot x|} \equiv \mathrm{e}^u$.



With this, the metric on $\mathcal{T}$ becomes

$$\mathrm{d}s_{\mathcal{T}}^2 = \mathrm{e}^{2u}\left(-\mathrm{d}u^2 + \mathrm{d}s_{H^3}^2\right) \ . \tag{4}$$

Now, $H^3 \cong \mathrm{SO}(1,3)/\mathrm{SO}(3)$ on account of following maps:

$$\begin{aligned} \alpha_{\mathcal{T}} &: \ \mathrm{SO}(1,3)/\mathrm{SO}(3) \to H^3 \ , \quad [\Lambda_{\mathcal{T}}] \mapsto y^\mu := (\Lambda_{\mathcal{T}})^\mu{}_0 \\ \alpha_{\mathcal{T}}^{-1} &: \ H^3 \to \mathrm{SO}(1,3)/\mathrm{SO}(3) \ , \quad y^\mu \mapsto [\Lambda_{\mathcal{T}}] \ , \end{aligned} \tag{5}$$

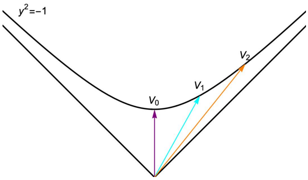
where the representative element (for right $\mathrm{SO}(3)$-multiplication)

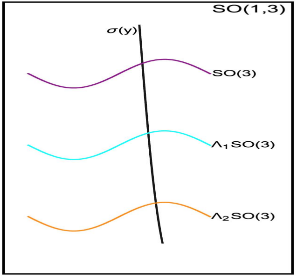
$$\Lambda_{\mathcal{T}} = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta} \\ \gamma\boldsymbol{\beta}^T & \mathbb{1} + (\gamma - 1)\frac{\boldsymbol{\beta}\otimes\boldsymbol{\beta}}{\boldsymbol{\beta}^2} \end{pmatrix} \ ; \ \beta^a = \frac{y^a}{y^0} \ , \ \gamma = \frac{1}{\sqrt{1 - \vec{\beta}^2}} \ . \tag{6}$$

This generic expression for the boost $\Lambda_{\mathcal{T}}$ can be obtained by exponentiation with boost generators K_a:

$$\Lambda_{\mathcal{T}} = \exp\left(\eta^a K_a\right) \ ; \quad \beta^a := \frac{\eta^a}{\sqrt{\vec{\eta}^2}} \tanh\sqrt{\vec{\eta}^2} \ . \tag{7}$$

Looking at the coset corresponence more pictorially:





- A given timelike vector V_i on the hyperbola can be brought to the vector $V_0 = (1,0,0,0)^T$ via a boost Λ_i.
- The vector V_i is uniquely associated with a coset $\Lambda_i\, SO(3)$ due to the fact that its stabilizer is nothing but $\Lambda_i\, SO(3)\, \Lambda_i^{-1}$.
- The hyperbola H^3 is the curve σ, parameterised by y, that passes through every coset once.
- Alternatively, H^3 is *the* orbit under the adjoint action of the Lorentz group $SO(1,3)$ on the coset space.

The lightcone exterior $\mathcal{S}$ can be foliated with de Sitter space dS_3

$$y \cdot y \equiv \eta_{\mu\nu}\, y^\mu\, y^\nu \;=\; 1\,. \tag{8}$$

To see the foliation explicitly we note down the following maps

$$\begin{aligned} \varphi_{\mathcal{S}} &: \;\mathbb{R} \times \mathrm{dS}_3 \to \mathcal{S}\,, \quad (u, y^\mu) \mapsto \mathrm{e}^u\, y^\mu =: x^\mu \\ \varphi_{\mathcal{S}}^{-1} &: \;\mathcal{S} \to \mathbb{R} \times \mathrm{dS}_3\,, \quad x^\mu \mapsto \left(\ln \sqrt{\eta_{\mu\nu} x^\mu x^\nu}, \frac{x^\mu}{\sqrt{\eta_{\mu\nu} x^\mu x^\nu}} \right)\,, \end{aligned} \tag{9}$$

where $\mathrm{e}^u = \sqrt{\eta_{\mu\nu} x^\mu x^\nu}$. This makes the flat metric on $\mathcal{S}$ conformal to a Euclidean cylinder $\mathbb{R} \times \mathrm{dS}_3$ with spatial parameter u:

$$\mathrm{d}s_{\mathcal{S}}^2 = e^{2u} \left(\mathrm{d}u^2 + \mathrm{d}s_{\mathrm{dS}_3}^2 \right)\,, \tag{10}$$

where $\mathrm{d}s_{\mathrm{dS}_3}^2$ is the metric on dS_3 induced from (8). Now, the equivalence between dS_3 and the coset $SO(1,3)/SO(1,2)$ is clear from

$$\begin{aligned} \alpha_{\mathcal{S}} &: \;SO(1,3)/SO(1,2) \to \mathrm{dS}_3\,, \quad [\Lambda] \mapsto \Lambda_{0\mu} =: y^\mu \\ \alpha_{\mathcal{S}}^{-1} &: \;\mathrm{dS}_3 \to SO(1,3)/SO(1,2)\,, \quad y^\mu \mapsto [\Lambda_{\mathcal{S}}]\,, \end{aligned} \tag{11}$$

where the representative element $\Lambda_{\mathcal{S}}$ is defined as follows

$$\Lambda_{\mathcal{S}} = \begin{pmatrix} 1+(\gamma-1)\frac{\beta_1^2}{\boldsymbol{\beta}^2} & -(\gamma-1)\frac{\beta_1\beta_2}{\boldsymbol{\beta}^2} & -(\gamma-1)\frac{\beta_1\beta_3}{\beta^2} & \beta_1\gamma \\ (\gamma-1)\frac{\beta_1\beta_2}{\boldsymbol{\beta}^2} & 1-(\gamma-1)\frac{\beta_2^2}{\boldsymbol{\beta}^2} & -(\gamma-1)\frac{\beta_2\beta_3}{\boldsymbol{\beta}^2} & \beta_2\gamma \\ (\gamma-1)\frac{\beta_1\beta_3}{\boldsymbol{\beta}^2} & -(\gamma-1)\frac{\beta_2\beta_3}{\boldsymbol{\beta}^2} & 1-(\gamma-1)\frac{\beta_3^2}{\boldsymbol{\beta}^2} & \beta_3\gamma \\ \beta_1\gamma & -\beta_2\gamma & -\beta_3\gamma & \gamma \end{pmatrix}$$

with (here we have choosen (0,0,0,1) as the base vector)

$$\beta_a = \frac{y^{a-1}}{y^3}\ ,\ \boldsymbol{\beta}^2 = -\eta_{ab}\beta^a\beta^b \geq 0 \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1-\boldsymbol{\beta}^2}}\ . \qquad (12)$$

Notice the presence of the 3-dimensional Minkowski metric $\eta_{ab} = \mathrm{diag}(-1,1,1)_{ab}$; it's because the stabilizer $SO(1,2)$ here is the isometry group of $\mathbb{R}^{1,2}$. Here again, one obtains $\Lambda_{\mathcal{S}}$, analogous to the previous case (7), by exponentiation with coset generators and using

$$\Lambda_{\mathcal{S}} = \exp(-\kappa_3 J_1 + \kappa_2 J_2 + \kappa_1 K_3)\ ;\ \beta_a := \frac{\kappa_a}{\sqrt{\boldsymbol{\kappa}^2}} \tan\sqrt{\boldsymbol{\kappa}^2}\ ;\ \ \boldsymbol{\kappa}^2 = -\eta^{ab}\kappa_a\kappa_b\ .$$

For reductive coset spaces G/H the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ with:

$$[I_A, I_B] \;=\; f_{AB}{}^C\, I_C \quad \text{where} \quad A, B, C \;=\; 1, \ldots, 6 \;, \tag{13}$$

splits into a Lie subalgebra $\mathfrak{h}$ and an orthogonal complement $\mathfrak{m}$ such that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ ($a = 1, 2, 3$ and $i = 4, 5, 6$):

$$\mathfrak{g} \;=\; \mathfrak{h} \oplus \mathfrak{m} \quad \Longrightarrow \quad \{I_A\} \;=\; \{I_i\} \cup \{I_a\} \;, \tag{14}$$

which, for symmetric spaces with $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, satisfy

$$[I_i, I_j] \;=\; f_{ij}{}^k I_k \;, \quad [I_i, I_a] \;=\; f_{ia}{}^b I_b \quad \text{and} \quad [I_a, I_b] \;=\; f_{ab}{}^i I_i \;. \tag{15}$$

The Cartan one-forms $e^A = g^{-1}\mathrm{d}g$ also splits into $\{e^i\} \cup \{e^a\}$ where $e^i = e^i_a\, e^a$ and they obey following structure equations:

$$\mathrm{d}e^a + f_{ib}{}^a\, e^i \wedge e^b \;=\; 0, \; \mathrm{d}e^i + \tfrac{1}{2} f_{jk}{}^i\, e^j \wedge e^k + \tfrac{1}{2} f_{ab}{}^i\, e^a \wedge e^b \;=\; 0 \;. \tag{16}$$

For the coset $\mathrm{SO}(1,3)/\mathrm{SO}(3)$ we have $I_i = J_i$ and $I_a = K_a$:

$$f_{ij}{}^{k} = \varepsilon_{i-3\,j-3\,k-3}\ ,\ \ f_{ia}{}^{b} = \varepsilon_{i-3\,a\,b} \quad \text{and} \quad f_{ab}{}^{i} = -\varepsilon_{a\,b\,i-3}\ . \tag{17}$$

The Maurer–Cartan one-forms $\Lambda_{\mathcal{T}}^{-1}\mathrm{d}\Lambda_{\mathcal{T}} = e^a I_a + e^i I_i$ are

$$e^a = \left(\delta^{ab} - \frac{y^a\, y^b}{y^0(1+y^0)}\right)\mathrm{d}y^b\ , \qquad e^i = \varepsilon_{i-3ab}\,\frac{y^a}{1+y^0}\,\mathrm{d}y^b\ . \tag{18}$$

Notice that $e^0 := \mathrm{d}u$ & e^a provides, locally, an orthonormal-frame on the cotangent bundle $T^*(U \subset \mathbb{R}\times H^3)$:

$$\mathrm{d}s^2_{cyl} = -\mathrm{d}u^2 + \mathrm{d}s^2_{H^3} = -e^0\otimes e^0 + \delta_{ab}\, e^a\otimes e^b\ . \tag{19}$$

They can be pulled back to Minkowski space $\mathbb{R}^{1,3}$ via $\varphi_{\mathcal{T}}$ (2).

For the coset $\mathrm{SO}(1,3)/\mathrm{SO}(1,2)$ we employ following generators:

$$I_i \in \{K_1, K_2, J_3\} \quad \text{and} \quad I_a \in \{J_1, J_2, K_3\} \tag{20}$$

that give rise to following set of structure coefficients

$$\begin{gathered} f_{ij}{}^{k} \;=\; \varepsilon_{i-3\,j-3\,k-3}\,(1-2\,\delta_{k6})\;, \quad f_{ab}{}^{i} \;=\; \varepsilon_{a\,b\,i-3}\;, \\ f_{ia}{}^{b} \;=\; \varepsilon_{i-3\,a\,b}\,(1-2\,\delta_{a3}) \end{gathered} \tag{21}$$

where the indices for the terms inside the bracket are not summed over. The Maurer–Cartan one-forms $\Lambda_{\mathcal{S}}^{-1}\,\mathrm{d}\Lambda_{\mathcal{S}} = e^a\,I_a + e^i\,I_i$ are

$$e^a = \mathrm{d}y^{3-a} - \frac{y^{3-a}}{1+y^3}\,\mathrm{d}y^3\;, \quad e^i = -\varepsilon_{i-3\,a\,b}\,\frac{y^{3-a}}{1+y^3}\,\mathrm{d}y^{3-b} \tag{22}$$

yielding the following metric on the cylinder $\mathbb{R}\times \mathrm{dS}_3$ (10)

$$\tilde{g} \;=\; e^0\otimes e^0 + \eta_{ab}\,e^a\otimes e^b\;; \quad \eta \;=\; (-,+,+)\;. \tag{23}$$

For a symmetric space, such as dS_3 & H^3, an $\mathrm{SO}(1,3)$-invariant[2] connection one-form $\mathcal{A}$ in "temporal" gauge $\mathcal{A}_0 = 0$ is given by

$$\mathcal{A} = e^i\, I_i + \phi(u) e^a\, I_a\ . \tag{24}$$

The field strength $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ becomes ($\dot{\phi} := \partial_u \phi$)

$$\mathcal{F} = \dot{\phi}\, I_a\, e^0 \wedge e^a + \frac{1}{2}(\phi^2 - 1)\, f_{ab}{}^i\, I_i\, e^a \wedge e^b\ . \tag{25}$$

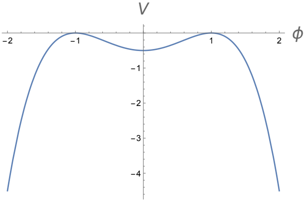
The corresponding Yang–Mills action, for both cases, simplifies to:

$$\begin{aligned} S_{YM} &= -\frac{1}{4g^2} \int_{\mathbb{R}\times H^3/\mathrm{dS}_3} \mathrm{tr}_{\mathrm{ad}}(\mathcal{F} \wedge *\mathcal{F}) \\ &= \frac{6}{g^2} \int_{\mathbb{R}\times H^3/\mathrm{dS}_3} \mathrm{dvol}\, \left(\frac{1}{2}\dot{\phi}^2 - V(\phi) \right)\ , \end{aligned} \tag{26}$$

[2] D. Kapetanakis & G. Zoupanos, *Phys. Rept.* **219** (1992) 1.

which describes a mechanical particle inside an inverted-double-well potential $V(\phi) = -\frac{1}{2}(\phi^2-1)^2$ with EOM:

$$\ddot{\phi} \;=\; 2\,\phi\,(\phi^2 - 1)\;. \qquad (27)$$



A generic solution $\phi_{\epsilon,u_0}(u)$, with energy $\epsilon = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ and 'time'-shift u_0, can be written using Jacobi elliptic functions:

$$\phi_{\epsilon,u_0}(u) \;=\; f_-(\epsilon)\,\mathrm{sn}\big(f_+(\epsilon)(u-u_0), k\big) \qquad (28)$$

where $f_\pm(\epsilon) = \sqrt{1 \pm \sqrt{-2\epsilon}}$ and $k^2 = \frac{f_-(\epsilon)}{f_+(\epsilon)}$. Special cases include:

$$\phi \;=\; \begin{cases} 0 & \text{for} \quad \epsilon = -\frac{1}{2} \\ \tanh(u-u_0) & \text{for} \quad \epsilon = 0 \\ \pm 1 & \text{for} \quad \epsilon = 0 \end{cases}\;. \qquad (29)$$

Pulling the orthonormal-frame (e^0, e^a) on $\mathbb{R} \times H^3$ back to $\mathcal{T}$ with the map $\varphi_{\mathcal{T}}$ (2) we get

$$\begin{aligned} e^0 := \mathrm{d}u \; &= \; \frac{t\,\mathrm{d}t - r\,\mathrm{d}r}{t^2 - r^2} \; ; \quad r := \sqrt{\vec{x}^{\,2}} \; , \\ e^a \; &= \; \frac{1}{|x|}\Big(\mathrm{d}x^a - \frac{x^a}{|x|}\,\mathrm{d}t + \frac{x^a}{|x|(|x|+t)}\,r\,\mathrm{d}r\Big) \; . \end{aligned} \tag{30}$$

The colour electric $E_i := F_{0i}$ and magnetic $B_i := \frac{1}{2}\varepsilon_{ijk}\,F_{jk}$ fields in terms of $\phi(x) = \phi(u(x))$ are given by

$$\begin{aligned} E_a &= \frac{1}{|x|^3}\Big\{(\phi^2{-}1)\,\varepsilon_{ab}{}^{i-3}\,x^b\,I_i - \dot{\phi}\Big(t\,\delta^{ab} - \frac{x^a\,x^b}{|x|+t}\Big)I_b\Big\} \; , \\ B_a &= -\frac{1}{|x|^3}\Big\{(\phi^2{-}1)\,\Big(t\,\delta^{a\,i-3} - \frac{x^a\,x^{i-3}}{|x|+t}\Big)I_i + \dot{\phi}\,\varepsilon_{ab}{}^{c}\,x^b\,I_c\Big\} \; . \end{aligned} \tag{31}$$

Similarly, one pulls the local orthonormal-frame on $\mathbb{R} \times \mathrm{dS}_3$ back to $\mathcal{S}$ and computes the corresponding color electric- and magnetic-fields. As it turns out, the stress-energy tensor

$$T_{\mu\nu} \;=\; -\tfrac{1}{2g^2}\,\mathrm{tr}_{\mathsf{ad}}\Big(F_{\mu\alpha}\,F_{\nu\beta}\,\eta^{\alpha\beta} - \tfrac{1}{4}\eta_{\mu\nu}F^2\Big)\;;\; F^2 = F_{\mu\nu}\,F^{\mu\nu}\;, \tag{32}$$

evaluates to the same expression, written compactly as follows

$$T_{\mu\nu} \;=\; \frac{\epsilon}{g^2}\,\frac{4\,x_\mu x_\nu - \eta_{\mu\nu}\,x{\cdot}x}{(x{\cdot}x)^3}\;, \tag{33}$$

which is singular on the lightcone! This, curiously, can be written as a pure “improvement” term:

$$T_{\mu\nu} \;=\; \partial^\rho S_{\rho\mu\nu} \qquad \text{with} \qquad S_{\rho\mu\nu} \;=\; \frac{\epsilon}{g^2}\,\frac{x_\rho\eta_{\mu\nu} - x_\mu\eta_{\rho\nu}}{(x{\cdot}x)^2}\;. \tag{34}$$

One attempt to regularize $T_{\mu\nu}$ is via a nonsingular $S^{\text{reg}}_{\rho\mu\nu}$ as follows

$$\begin{aligned} S^{\text{reg}}_{\rho\mu\nu} &= \frac{\epsilon}{g^2}\,\frac{x_\rho \eta_{\mu\nu} - x_\mu \eta_{\rho\nu}}{(x{\cdot}x+\delta)^2} \\ \Rightarrow T^{\text{reg}}_{\mu\nu} &= \frac{\epsilon}{g^2}\,\frac{4\,x_\mu x_\nu - \eta_{\mu\nu} x{\cdot}x + 3\,\delta\,\eta_{\mu\nu}}{(x{\cdot}x+\delta)^3}\;, \end{aligned} \tag{35}$$

which yields vanishing energy and momenta for any finite value of the regularization parameter δ (fall-off at spatial infinity is fast enough).

Another way to regularize $T_{\mu\nu}$ is to directly shift the denominator by δ so that (up to an equivalence with above improvement term):

$$T^{\delta}_{\mu\nu} = \frac{\epsilon}{g^2}\,\frac{4\,x_\mu x_\nu - \eta_{\mu\nu} x{\cdot}x}{(x{\cdot}x+\delta)^3} \sim \frac{\epsilon}{g^2}\,\frac{-3\,\delta\,\eta_{\mu\nu}}{(x{\cdot}x+\delta)^3}\;, \tag{36}$$

which is regular in the entire Minkowski space!

The null (upper $\mathcal{L}_+$ or lower $\mathcal{L}_-$) hypersurfaces are isomorphic to $\mathrm{SO}(1,3)/\mathrm{ISO}(2)$ where the stability subgroup $\mathrm{ISO}(2) = \mathrm{E}(2)$ is spanned by two translations and one rotation generators:

$$\begin{aligned} \mathcal{L}_+ \quad &\leftrightarrow \quad I_a \in \{P_1, P_2, K_3\} \quad \text{and} \quad I_i \in \{P_3, P_4, J_3\} \\ \mathcal{L}_- \quad &\leftrightarrow \quad I_a \in \{P_3, P_4, K_3\} \quad \text{and} \quad I_i \in \{P_1, P_2, J_3\} \end{aligned} \tag{37}$$

where $P_1 := K_1 + J_2$, $P_2 := K_2 - J_1$, $P_3 := K_1 - J_2$ & $P_4 := K_2 + J_1$. Notice that $\mathrm{SO}(1,3)/\mathrm{ISO}(2)$ is not reductive. In fact, $\mathfrak{m}$ is another $\mathfrak{g}$-subalgebra and is not an orthogonal complement. Computing Maurer–Cartan one-forms (only e^a exists) on $\mathcal{L}_+$ we find that

$$\mathrm{d}s^2_{\mathbb{R}^{1,3}}\Big|_{\mathcal{L}_+} = 4\, e^1 \otimes e^1 + 4\, e^2 \otimes e^2 \ , \tag{38}$$

i.e. the metric is degenerate as expected. More importantly, there is no folitation of the lightcone here and hence no dynamics!

Summary and outlook

- We obtained new YM solutions on the interior resp. exterior of lightcone with gauge group $G = \mathrm{SO}(1,3)$ by employing coset foliation with stabilizer subgroup $\mathrm{SO}(3)$ resp. $\mathrm{SO}(1,2)$.
- Although the fields and their stress-energy tensor is singular at lightcone, the latter can nevertheless be regularized.
- We have non-reductive coset $\mathrm{SO}(1,3)/\mathrm{ISO}(2)$ for null (past/future) hypersurfaces, but there is no foliation – no YM solutions here.
- In ongoing/future works we plan to study Yang–Mills dynamics on other related/unrelated coset spaces like $\mathrm{SO}(2,2)/\mathrm{SO}(1,2)$ and $\mathrm{G}_2/\mathrm{SU}(3)$.

Thank You!

Questions?