Spectral distances on double Moyal plane

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Why non-commutative spaces?

- In 1999, Witten along with Sieberg demostrated that in certain low energy regime of string theory, non-commutativity of Moyal type crops up.
- Non-commutative geometry (NCG) as developed by Alain Connes has been successfully applied to many problems of physical interest like quantum Hall effect, Zitterbewegung and is a promising candidate for quantum gravity.
- Connes alongwith others have been successful in reformulating standard model based on NCG through "almost-commutative" manifold and has provided a remarkable geometrical interpretation for the Higgs field.

Moyal plane: Hilbert-Schmidt formalism

We begin with the non-commutative algebra

$$[\hat{x}_\alpha, \hat{x}_\beta] = i\theta_{\alpha\beta} = i\theta\epsilon_{\alpha\beta} \; ; \; \alpha, \beta \in \{1, 2\}, \tag{1}$$

which along with

$$[\hat{x}_\alpha, \hat{p}_\beta] = i\hbar\delta_{\alpha\beta} \quad ; \quad [\hat{p}_\alpha, \hat{p}_\beta] = 0, \tag{2}$$

gives the modified Hiesenberg algebra. This renders the configuration space "fuzzy" due to the uncertainty relation:

$$(\Delta x_1)(\Delta x_2) \geq \frac{\theta}{2}. \tag{3}$$

Now one can define following ladder operators

$$\hat{b} = \frac{1}{\sqrt{2\theta}} (\hat{x}_1 + i\hat{x}_2) \; ; \; \hat{b}^\dagger = \frac{1}{\sqrt{2\theta}} (\hat{x}_1 - i\hat{x}_2) \tag{4}$$

satisfying $[\hat{b}, \hat{b}^\dagger] = 1$ and $\hat{b}\,|0\rangle = 0$.

Moyal plane: Hilbert-Schmidt formalism

One can then naturally associate a Hilbert space $\mathcal{H}_c$ to the coordiate algebra (1), which is isomorphic to the Hilbert space of 1D qunatum mechanical SHO.

$$\mathcal{H}_c = \mathrm{span}\left\{ |n\rangle = \frac{1}{\sqrt{n!}}(\hat{b}^\dagger)^n |0\rangle \right\}_{n=0}^{\infty} \tag{5}$$

This replaces the 2D configuration space of commutative QM. The quantum Hilbert space $\mathcal{H}_q$ then comes automatically by considering the set of Hilbert-Schmidt operators on $\mathcal{H}_c$ (which again forms a Hilbert space) as

$$\mathcal{H}_q = \{\psi : \psi \in \mathcal{B}(\mathcal{H}_c), Tr_c(\psi^\dagger \psi) < \infty\}. \tag{6}$$

Also the associated inner product (and thus norm) can easily be defined by

$$(\phi|\psi) = Tr_c(\phi^\dagger \psi). \tag{7}$$

Moyal plane: Hilbert-Schmidt formalism

The next step is to look for a unitary representation $\hat{X}_i$ and $\hat{P}_i$ of position $\hat{x}_i$ and canonical momenta $\hat{p}_i$ respectively. It turns out that, following representation does the job.

$$\hat{X}_i\psi = \hat{x}_i\psi, \quad \hat{P}_i\psi = \frac{1}{\theta}\epsilon_{ij}[\hat{x}_j,\psi] = \frac{1}{\theta}\epsilon_{ij}\left(\hat{X}_j^L - \hat{X}_j^R\right)\psi \quad (8)$$

Here the left/right action for any $\psi \in \mathcal{H}_q$ is defined as

$$\hat{X}_i^L\psi \equiv \hat{X}_i\psi = \hat{x}_i\psi \ ; \ \hat{X}_i^R\psi = \psi\hat{x}_i \ : \ \forall \ \psi = \sum_{m,n=0}^{\infty} C_{mn}\left|m\right\rangle\left\langle n\right|$$

These operators respect the modified Heisengberg algebra i.e.

$$[\hat{X}_i^L, \hat{X}_j^L] \equiv [\hat{X}_i, \hat{X}_j] = i\theta\epsilon_{ij} \ ; \ [\hat{X}_i, \hat{P}_j] = i\delta_{ij} \ ; \ [\hat{P}_i, \hat{P}_j] = 0 \quad (9)$$

Using this prescription one can work out various QM problem in this space (e.g. see [1])

Connes spectral distance

How to study the geometry of such NC spaces where the notion of points, lines etc. breaks down?
NCG provides a way out a set of data called "spectral triple" $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, which encodes the geometrical information of generalized spaces.

- $\mathcal{H}$: The Hilbert space.
- $\mathcal{A}$: An involutive C^* algebra (satisfying $\|a^*a\| = \|aa^*\| = \|a\|^2$), with some faithful representation (π) as bounded operators on $\mathcal{H}$.
- $\mathcal{D}$: A self-adjoint operator on $\mathcal{H}$ called generalized Dirac operator.

The primary motivation for such a construction came from the celebrated theorem by Gelfand and Naimark which establishes duality between (resp. catagories of) C^* Algebra and Topological spaces.

Connes spectral distance

The role of Dirac operator is to define the neccessary differential structure on the manofold while the link between algebra $\mathcal{A}$ and Hilbert space $\mathcal{H}$ is provided by following isometric embedding

$$\pi(\mathcal{A}) \hookrightarrow \mathcal{B}(\mathcal{H}). \tag{10}$$

Then, one defines *States* ω as positive linear functionals of norm 1 over $\mathcal{A}$. These are the analogue of points in usual commutative geometry. Distance between any two states (ω, ω') are given by

$$\begin{aligned} d(\omega, \omega') &= \sup_{a \in B} |\omega(a) - \omega'(a)|, \\ B &= \{a \in \mathcal{A} : \|[\mathcal{D}, \pi(a)]\|_{op} \leq 1\}, \\ \|\mathcal{O}\|_{op} &= \sup_{\phi \in \mathcal{H}} \frac{\|\mathcal{O}\phi\|}{\|\phi\|}, \end{aligned} \tag{11}$$

Spectral distance: Two point space

It's an abstract space of two c-numbers with spectral triple $\left(\mathcal{A}_2 = \mathbb{C}^2, \mathcal{H}_2 = \mathbb{C}^2, \mathcal{D}_2 = \begin{pmatrix} 0 & \Lambda \\ \bar{\Lambda} & 0 \end{pmatrix}\right)$ where Λ is a constant complex parameter of length-inverse dimension. Any algebra element can be written in both the canonical basis $\text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, or the 2×2 matrix basis $\text{Span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (\doteq \omega_1), \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} (\doteq \omega_2) \right\}$. After calculation one gets (see [2])

$$\boxed{d_{D_F}(\omega_1, \omega_2) = \frac{1}{\Lambda}}$$

Spectral distance: Moyal plane

Here the spectral triple is $\mathcal{A}_M := \mathcal{H}_q$, $\mathcal{H}_M := \mathcal{H}_c \otimes \mathbb{C}^2$ and $\mathcal{D}_M = \hat{P}_1\sigma_1 + \hat{P}_2\sigma_2 = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & b^\dagger \\ b & 0 \end{pmatrix}$

The elements $a \in \mathcal{A}$ acts on the elements $\psi = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix} \in \mathcal{H}$ through diagonal representation $\pi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. Then one can calculate the spectral distance between pure states $\rho_z = |z\rangle\langle z|$ and $\rho_0 = |0\rangle\langle 0|$, where the coherent state basis $|z\rangle = e^{-\hat{b}\bar{z}+\hat{b}^\dagger z}|0\rangle$ with $z = \frac{x_1+ix_2}{\sqrt{2\theta}}$ are the eigen states of $\hat{b}$, to get

$$\boxed{d_{D_M}(\rho_z, \rho_0) = \sqrt{2\theta}|z|}$$

Double Moyal plane

Double Moyal plane is the (tensor) product space of Moyal plane and the two point space. By "doublying procedure", it's spectral triple is obtained as

$$\mathcal{A}_t = \mathcal{A}_M \otimes \mathcal{A}_2 = \mathcal{H}_q \otimes \mathbb{C}^2 \; ; \; \mathcal{H}_t = \mathcal{H}_M \otimes \mathcal{H}_2 = (\mathcal{H}_c \otimes \mathbb{C}^2) \otimes \mathbb{C}^2$$

$$\mathcal{D}_t = \mathcal{D}_M \otimes \mathbb{1} + \gamma \otimes \mathcal{D}_2$$

where $\gamma = \sigma_3$ for Moyal plane. It turns out that the Dirac eigen spinors are the natural basis for distance calculation, and for Moyal plane they are given by

$$|0\rangle\rangle := \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix}, |m\rangle\rangle_\pm := \frac{1}{\sqrt{2}} \begin{pmatrix} |m\rangle \\ \pm |m-1\rangle \end{pmatrix} \quad ; \quad m = 1, 2, 3, \cdots$$

They furnish furnish a complete and orthonormal basis for $\mathcal{H}_M$ and the corresponding eigenvalues λ_m are,

$$\lambda_0 = 0 \quad ; \quad \lambda_m^\pm = \pm\sqrt{\frac{2m}{\theta}}$$

Double Moyal plane

Taking hint from the Moyal plane, we construct following eigen spinors (for $m = 0$) for the Dirac operator $\mathcal{D}_t$, with eigen values $\lambda_{\pm}^{(m)} = \pm|\Lambda|\sqrt{\kappa m + 1}$:

$$\left|\Psi_{\pm}^{(m)}\right\rangle = N_m\Big[V_{++}^{(m)} + V_{--}^{(m)} \pm V_{-+}^{(m)}\left(\sqrt{\kappa m + 1} \mp \sqrt{\kappa m}\right) \mp V_{+-}^{(m)}\left(\sqrt{\kappa m + 1} \pm \sqrt{\kappa m}\right)\Big]; \;\; \kappa = \frac{2}{\theta \Lambda^2}$$

$$\left|\widetilde{\Psi}_{\pm}^{(m)}\right\rangle = N_m\Big[V_{+-}^{(m)} + V_{-+}^{(m)} \pm V_{++}^{(m)}\left(\sqrt{\kappa m + 1} \pm \sqrt{\kappa m}\right) \mp V_{--}^{(m)}\left(\sqrt{\kappa m + 1} \mp \sqrt{\kappa m}\right)\Big],$$

where the basis elements are given by

$$V_{\pm\pm}^{(m)} = \begin{pmatrix} |m\rangle \\ \pm\,|m-1\rangle \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix},$$

and the normalization factor $N_m = \frac{1}{\sqrt{m\kappa + 1}}$

Distances on double Moyal plane

For $m = 0$ we obtain following eigen spinors

$$\left|\Psi_{\pm}^{(0)}\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}|0\rangle \\ 0\end{pmatrix} \otimes \begin{pmatrix}1 \\ \pm 1\end{pmatrix} \tag{12}$$

There are three different kinds of distances in double Moyal plane as shown in following figure. Here pure states are given by $\phi = \rho_z \otimes \omega_i$; $i = 1, 2$

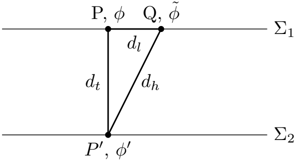

Figure: $M \cup M$, Space associated with doubly spectral triple.

Transverse distance

Here we consider states $\Omega_z^{(1)} = |z\rangle\langle z| \otimes \omega_1$ on Moyal plane Σ_1 and it's clone $\Omega_z^{(2)} = |z\rangle\langle z| \otimes \omega_2$ on Moyal plane Σ_2. On top of this, we work with an algebra element $a_t = \mathbb{1}_{\mathcal{H}_q} \otimes a_2$ ($a_2 \in \mathcal{A}_2$) with representation

$$\pi(a_t) = \begin{pmatrix} \mathbb{1}_{\mathcal{H}_q} & 0 \\ 0 & \mathbb{1}_{\mathcal{H}_q} \end{pmatrix} \otimes \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}. \tag{13}$$

Then the distance formula (11) becomes

$$d_t\left(\Omega_z^{(1)}, \Omega_z^{(2)}\right) = \sup_{a_t \in B_T} |\mathrm{Tr}_M\left(d\Omega_z a_t\right)| = \sup_{a_t \in B_T} |c_1 - c_2| \tag{14}$$

With this choice the Lipschitz ball condition turns out to be $|\Lambda(c_1 - c_2)| \leq 1$, producing the distance on two point space i.e.

$$d_t\left(\Omega_z^{(1)}, \Omega_z^{(2)}\right) = \frac{1}{|\Lambda|} \tag{15}$$

Transverse distance

However, the Moyal plane algebra is non-unital i.e. $\mathbb{1}_{\mathcal{H}_q} \notin \mathcal{A}_M = \mathcal{H}_q$. To get around this problem we consider finite order projection operator $\mathcal{P}_N$ on $\mathcal{H}_t$ constructed using the Dirac eigen spinors. Upon calculation this projection operator splits as

$$\mathcal{P}_N = \begin{pmatrix} P_N & 0 \\ 0 & P_{N-1} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{16}$$

where $P_N := |0\rangle\langle 0| + |1\rangle\langle 1| + ... + |N\rangle\langle N|$ are the N-th order projection operator in $\mathcal{H}_c$. So instead of working with $\mathbb{1}_{\mathcal{H}_q}$, we choose to work with P_N and P_{N-1} as diagonal entries in the left slot of $\pi(a_t)$. The form of the distance (15) remains valid as we keep on increasing the order of projection from $N = 2$ onwards.

Longitudinal distance

Here we compute distance between states $\Omega_{dz}^{(i)} = \rho_{dz} \otimes \omega_i$ and $\Omega_0^{(i)} = \rho_0 \otimes \omega_i$. Using a general algebra element $a_l = a \otimes a_2$ $(a \in \mathcal{A}_M)$ and after little manipulation we obtain

$$d_l\left(\Omega_{dz}^{(i)}, \Omega_0^{(i)}\right) = \sup_{a_l \in B_T} |c_i| \cdot |\rho_{dz}(a) - \rho_0(a)| \qquad (17)$$

Also the total ball condition becomes

$$[\mathcal{D}_T, \pi(a_T)] = [\mathcal{D}_M, \pi(a)] \otimes a_2 + a(c_1 - c_2)\sigma_3 \otimes \mathcal{D}_2 \qquad (18)$$

By symmetry argument we demand $|c_1| = |c_2|$ or $c_1 = \pm c_2$ $(c_i \in \mathbb{R})$. The condition $c_1 = c_2$ renders the total ball condition to be same as that of Moyal plane alone producing

$$d_l\left(\Omega_{dz}^{(i)}, \Omega_0^{(i)}\right) = \sqrt{2\theta}|dz| \qquad (19)$$

Hypotenuse distance

Also by using $a_l = \left(b + b^\dagger\right) \otimes a_2 \in \mathcal{A}_t$, inspired by our previous work (see [2]), we verified that the case of $c_1 = -c_2$ produces a distance which is lower than (19) and thus we discard this case.

For hypotenuse case we work with states $\Omega^{(1)}_{dz} = |dz\rangle\langle dz| \otimes \omega_1$ and $\Omega^{(2)}_0 = \rho_0 \otimes \omega_2$ and the algebra element $a_h = a_t + a_l$ with representation

$$\pi(a_h) = \begin{pmatrix} P_N & 0 \\ 0 & P_{N-1} \end{pmatrix} \otimes \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} + \begin{pmatrix} \left(b + b^\dagger\right) & 0 \\ 0 & \left(b + b^\dagger\right) \end{pmatrix} \otimes \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

This turns argument of supremum in (11) into

$$d_h\left(\Omega^{(1)}_{dz}, \Omega^{(2)}_0\right) = \sup_{a^{(h)}_S \in B_T} \left|2|dz|\alpha + (c_1 - c_2)\right| \qquad (20)$$

Hypotenuse distance

To get the ball condition we use the matrix representation of $[\mathcal{D}_T, \mathcal{P}_N \pi(a_h) \mathcal{P}_N]$ in Dirac eigen spinor basis (starting from $N = 1$) and employ C^* algebra identity to get

$$\Lambda\sqrt{\kappa X^2 + Y^2} \leq 1 \quad \forall \kappa \in \mathbb{Z}_+ \tag{21}$$

Surprisingly enough *Mathematica* could produce such simple result only for integer values of κ suggesting some quantization is taking place. After solving this optimization problem we obtain following Pythagoras equality proved earlier by Martinetti et. al. (see [3])

$$\begin{aligned} d_h\left(\Omega_{dz}^{(1)}, \Omega_0^{(2)}\right) &= \sqrt{2\theta|dz|^2 + \frac{1}{|\Lambda|^2}} \\ &= \sqrt{\left(d\left(\Omega_{dz}^{(i)}, \Omega_0^{(i)}\right)\right)^2 + \left(d_t\left(\Omega_z^{(1)}, \Omega_z^{(2)}\right)\right)^2} \end{aligned} \tag{22}$$

Summary and future goals

- Takeaway
  - A nice symmetric form of the Dirac eigen spinors for double Moyal plane has been obtained.
  - Three kinds of distances viz. transverse, longitudinal and hypotenuse has been computed verifying the Pythagoras theorem.
  - Some kind of quantization of the dimensionless quantity $\kappa = \frac{2}{\theta|\Lambda|^2}$ is taking place.
- What lies ahead...
  - Tackle the problem of 'time' in NCG by formulating it's Lorentzian version and study the issue of causality.
  - As a long term goal, we would like to put guage fields on such NC spaces and quantize them, starting from U(1) field.

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