Connes spectral distance on non-commutative spaces: fuzzy sphere & double Moyal plane

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- Incompatibility of ‘quantum mechanical’ particles and ‘classical’ background of general relativity
- Quantization of space time: a way out to many problems
- Unification of GR with standard model in the framework of “almost-commutative” space-time
- Open problem of quantization on fully non-commutative spaces

Two interesting 'toy' models: Moyal plane and fuzzy sphere

- Moyal plane

$$[\hat{x}_1, \hat{x}_2] = i\theta; \quad [\hat{x}_i, \hat{p}_j] = i\delta_{ij} \quad ; \quad [\hat{p}_i, \hat{p}_j] = 0 \qquad (1)$$

gives rise to hierarchy of Hilbert spaces

$$\mathcal{H}_c = \mathrm{Span}\{|n\rangle\}_{n=0}^{\infty} \quad ; \quad \mathcal{H}_q = \{\psi : \mathrm{tr}_c(\psi^\dagger \psi) < \infty\} \qquad (2)$$

where $\psi \equiv |n\rangle\langle n| \in \mathcal{H}_q$

Two interesting ‘toy’ models: Moyal plane and fuzzy sphere

- Fuzzy sphere

$$[\hat{x}_i, \hat{x}_j] = i\lambda\epsilon_{ijk}\hat{x}_k \quad (3)$$

Using Jordan-Schwinger map (3) can transormed as:

$$\hat{x}_i = \hat{\chi}^\dagger\sigma_i\hat{\chi} = \hat{\chi}^\dagger_\alpha\sigma_i^{\alpha\beta}\hat{\chi}_\beta \quad (4)$$

$$[\hat{\chi}_\alpha, \hat{\chi}^\dagger_\beta] = \frac{1}{2}\lambda\delta_{\alpha\beta}\ , \ \ [\hat{\chi}_\alpha, \hat{\chi}_\beta] = 0 = [\hat{\chi}^\dagger_\alpha, \hat{\chi}^\dagger_\beta]\ ; \qquad \alpha, \beta = 1, 2.$$

fuzzy sphere...

using $|n, n_3\rangle$ we get the configuration space

$$\mathcal{F}_c = \text{Span}\{|n, n_3\rangle \big| \forall n \in \mathbb{Z}/2, \ -n \leq n_3 \leq n\} \qquad (5)$$

each n corresponding to a fixed sphere of radius $r_n = \lambda\sqrt{n(n+1)}$, while $\mathcal{H}_q$ become

$$\mathcal{H}_q = \{\Psi \in Span\{|n, n_3\rangle\langle n, n_3'|\} : \ \ \text{tr}_c(\Psi^\dagger \Psi) < \infty\} = \bigoplus_n \mathcal{H}_n \qquad (6)$$

geometrical properties of general spaces: spectral triple

- Gelfand and Naimark's duality between Hilbert space and the space of bounded trace class operators.

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geometrical properties of general spaces: spectral triple

- Gelfand and Naimark's duality between Hilbert space and the space of bounded trace class operators.
- Alain Connes took this idea further to define spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ to extract geometrical information of the space.
- $\mathcal{H}$ is the Hilbert space, $\mathcal{A}$ is an involutive C^* algebra (acting on $\mathcal{H}$ through some representation say π), and $\mathcal{D}$ is the Dirac operator.

Connes Distance formula

States ω are positive linear functionals of norm 1 over $\mathcal{A}$.
Distance between any two pure states (ω, ω') are given by:

$$d(\omega, \omega') = \sup_{a \in B} |\omega(a) - \omega'(a)|,$$
$$B = \{a \in \mathcal{A} : \|[\mathcal{D}, \pi(a)]\|_{op} \leq 1\},$$
$$\|\mathcal{A}\|_{op} = \sup_{\phi \in \mathcal{H}} \frac{\|\mathcal{A}\phi\|}{\|\phi\|}. \tag{7}$$

after some modification it can be recasted as:

$$d(\rho, \rho + \Delta\rho) = \frac{\|\Delta\rho\|_{tr}^2}{inf\|[\mathcal{D}, \pi(\Delta\rho)] + \lambda[\mathcal{D}, \pi(\Delta\rho_\perp)]\|_{op}} \tag{8}$$

where $tr(\Delta\rho\Delta\rho_\perp) = 0$

fuzzy sphere: n=1

The spectral triple consists of the algebra $\mathcal{A} \equiv \mathcal{H}_q \ni |m\rangle\langle n|$, the Hilbert space $\mathcal{H} \equiv \mathcal{H}_c \otimes \mathbb{C}^2 \ni \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix}$, and the Dirac operator

$$\mathcal{D} \equiv \frac{1}{r_n} \hat{\vec{J}} \otimes \vec{\sigma} = \frac{1}{r_n} \begin{pmatrix} \hat{J}_3 & \hat{J}_- \\ \hat{J}_+ & -\hat{J}_3 \end{pmatrix} \tag{9}$$

Dirac eigen spinors for fuzzy sphere are given by:

$$|n_3\rangle\rangle_+ := \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{n_3+2}|n_3\rangle \\ \sqrt{1-n_3}|n_3+1\rangle \end{pmatrix} ; \; \mathcal{D}|n_3\rangle\rangle_+ = \frac{1}{r_1}|n_3\rangle\rangle_+$$
$$|n, n_3'\rangle\rangle_- := \frac{1}{\sqrt{3}} \begin{pmatrix} -\sqrt{1-n_3'}|n_3'\rangle \\ \sqrt{n_3'+2}|n_3'\rangle \end{pmatrix} ; \; \mathcal{D}|n, n_3'\rangle\rangle_- = -\frac{2}{r_1}|n, n_3'\rangle\rangle_- \tag{10}$$

fuzzy sphere: n=1

After some calculations and using C^* algebra property $||[\mathcal{D},\pi(a)]^\dagger[\mathcal{D},\pi(a)]||_{op} = ||[\mathcal{D},\pi(a)]||_{op}^2$ we get $||[\mathcal{D},\pi(a)]||_{op} = \frac{1}{r_1}\sqrt{max\{E_1,E_2\}}$ where E_1 and E_2 are eigen values of following matrix.

$$\begin{pmatrix} \begin{array}{l} 3|a_{0,1}|^2 + 2(a_{0,0}-a_{1,1})^2 \\ +|a_{0,-1}-2a_{1,0}|^2 + 6|a_{1,-1}|^2 \end{array} & \begin{array}{r} \sqrt{2}\{3a_{1,-1}(a_{0,1}+a_{-1,0}) \\ +(a_{0,0}-a_{1,1})(2a_{0,-1}-a_{1,0}) \\ +(a_{0,0}-a_{-1,-1})(2a_{1,0}-a_{0,-1})\} \end{array} \\ \begin{array}{l} \sqrt{2}\{3a_{-1,1}(a_{1,0}+a_{0,-1}) \\ +(a_{0,0}-a_{1,1})(2a_{-1,0}-a_{0,1}) \\ +(a_{0,0}-a_{-1,-1})(2a_{0,1}-a_{-1,0})\} \end{array} & \begin{array}{l} 3|a_{0,-1}|^2 + 2(a_{0,0}-a_{-1,-1})^2 \\ +|a_{1,0}-2a_{0,-1}|^2 + 6|a_{1,-1}|^2 \end{array} \end{pmatrix}$$

where the algebra element $a = \Delta\rho + \lambda\Delta\rho_\perp$ and $a_{mn} = \langle n|a|m\rangle$

Constructing $\Delta\rho$

Now starting from the pure state $\rho_0 = |1\rangle\langle 1|$ i.e. the north pole (N) of the n=1 fuzzy sphere and subjecting it to a unitary transformation with $\hat{U} \equiv e^{i\theta \hat{J}_2}$ we construct $\Delta\rho$ as:

$$\Delta\rho = \rho_\theta - \rho_0 = \hat{U}|1\rangle\langle 1|\hat{U}^\dagger - |1\rangle\langle 1|$$
$$= \begin{pmatrix} \cos^4\frac{\theta}{2} - 1 & -\frac{1}{\sqrt{2}}\sin\theta\cos^2\theta & \sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} \\ -\frac{1}{\sqrt{2}}\sin\theta\cos^2\theta & \frac{1}{2}\sin^2\theta & -\frac{1}{\sqrt{2}}\sin\theta\sin^2\theta \\ \sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} & -\frac{1}{\sqrt{2}}\sin\theta\sin^2\theta & \sin^4\frac{\theta}{2} \end{pmatrix}$$

This will however only give us the lower bound distance i.e. $\lambda = 0$ case.

$\Delta\rho_\perp$: infinitesimal case

After working with several choices of $\Delta\rho_\perp$ our search terminated with following most general form:

$$d\rho_\perp = \sum C_{ij}|i\rangle\langle j| \quad ; \quad i,j \in \{-1, 0, +1\} \tag{11}$$

where $C_{ij} = C_{ji}^*$ because of the hermiticity of $d\rho_\perp$. It can be written in matrix form as well:

$$d\rho_\perp = \begin{pmatrix} \mu_1 & i\alpha_1 & \gamma \\ -i\alpha_1 & \mu_0 & \beta \\ \gamma^* & \beta^* & \mu_{-1} \end{pmatrix} \ ; \ \mu_1, \mu_0, \mu_{-1}, \alpha_1 \in \mathbb{R} \text{ and } \beta, \gamma \in \mathbb{C}. \tag{12}$$

The result comes out to be:

$$d(\rho_0, \rho_{d\theta}) = r_1 \frac{d\theta}{\sqrt{2}} \tag{13}$$

$\Delta\rho_\perp$: for finite θ

- Here the lower bound itself gives $d(N, S) = \sqrt{2}\, r_1$, which matches exactly a result for distance between discrete states.

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- We therefore expect no contribution from $\Delta\rho_\perp$ in this case, which indeed happens for many constructions of such $\Delta\rho_\perp$.
- After going through many choices of the tranverse component, we again end up with the most general one which gives the best estimate.

most general $\Delta\rho_\perp$ for finite θ

$$\Delta\rho_\perp = \hat{U}\big(\mu_1|1\rangle\langle 1| + \mu_0|0\rangle\langle 0| + \mu_{-1}|-1\rangle\langle -1| + \alpha|1\rangle\langle 0| + \alpha^*|0\rangle\langle 1| + \gamma|1\rangle\langle -1| + \gamma^*|-1\rangle\langle 1| + \beta|0\rangle\langle -1| + |-1\rangle\langle 0|\big)\hat{U}^\dagger$$

This $\Delta\rho_\perp$ improves the distance corresponding to $\theta = \frac{\pi}{2}$ i.e. from North pole (N) to any point on th eequator (E) from the lower bound value $d(N,E) = 0.699 r_1$ to $d(N,E) = r_1$.

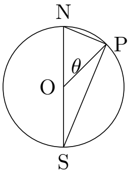


Figure: deviation from Pythagorean equality: $\sigma = NP^2 + PS^2 - NS^2$

most general $\Delta\rho_\perp$ for finite θ

Pythagorean equality arises for $n = \frac{1}{2}$ fuzzy sphere and we expect deviation as n increases. Following table illustrates our claim.

$(\theta^0, 180^0 - \theta^0)$	$\|\sigma\|$ $(*10^{-6})$
(10,170)	9.11
(20,160)	3.20
(30,150)	9.03
(40,140)	3.78
(50,130)	1.10
(60,120)	0
(70,110)	3.04
(80,100)	8.16
(90,90)	0

Double Moyal Plane

- Spectral triple of Moyal plane consists of $\mathcal{A} := \mathcal{H}_q$, $\mathcal{H} := \mathcal{H}_c \otimes \mathbb{C}^2$ and $\mathcal{D} = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & b^\dagger \\ b & 0 \end{pmatrix}$
- The elements $a \in \mathcal{A}$ acts on the elements $\psi = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix} \in \mathcal{H}$ through diagonal representation $\pi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$
- Double Moyal plane is the (tensor) product space of Moyal plane and (abstract) two point space whose spectral is $\left(\mathcal{A} = \mathbb{C}^2, \mathcal{H} = \mathbb{C}^2, \mathcal{D} = \begin{pmatrix} 0 & \Lambda \\ \bar{\Lambda} & 0 \end{pmatrix}\right)$; $\Lambda \in \mathbb{C}$

Double Moyal Plane: spectral triple

Doubling procedure results into following spectral triple for double Moyal plane:
$\mathcal{A}_t = \mathcal{H}_q \otimes \mathbb{C}^2$; $\mathcal{H}_t = (\mathcal{H}_c \otimes \mathbb{C}^2) \otimes \mathbb{C}^2 = \mathcal{H} \otimes \mathbb{C}^2$; $\mathcal{D}_t = \mathcal{D}_M \otimes \mathbb{1} + \gamma \otimes \mathcal{D}_F$
Where $\gamma = \sigma_3$ for Moyal plane. The Dirac operator $\mathcal{D}_t$ admits following eigen spinors with eigen values $\lambda^m_\pm = \pm|\Lambda|\sqrt{\kappa m + 1}$:

$$\left|m, \frac{1}{2}\right)_\pm = N_m[u^m_{++} + u^m_{--} \pm u^m_{-+}(\sqrt{\kappa m + 1} \mp \sqrt{\kappa m})$$
$$\mp u^m_{+-}(\sqrt{\kappa m + 1} \pm \sqrt{\kappa m})]; \;\; \kappa = \frac{2}{\theta \Lambda^2}$$
$$\left|m, -\frac{1}{2}\right)_\pm = N_m[u^m_{+-} + u^m_{-+} \pm u^m_{++}(\sqrt{\kappa m + 1} \pm \sqrt{\kappa m})$$
$$\mp u^m_{--}(\sqrt{\kappa m + 1} \mp \sqrt{\kappa m})]$$

Double Moyal Plane: Dirac eigen spinors

where the basis elements are given by:

$$u^m_{\pm\pm} = \begin{pmatrix} |m\rangle \\ \pm\,|m-1\rangle \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

There are three different kinds of distances in double Moyal plane as shown in following figure. Here pure states $\phi = \rho \otimes \omega$

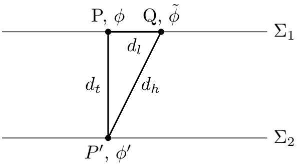


Figure: $M \cup M$, Space associated with doubly spectral triple.

Double Moyal Plane: longitudinal distane

In the above mentioned spinorial basis the longitudinal distances between states $\phi = |0\rangle\langle 0|$ and $\phi + d\phi = |0\rangle\langle 0| + d\bar{z}|0\rangle\langle 1| + dz|1\rangle\langle 0|$ comes out to be different on two copies of Moyal planes which is completely unacceptable. We therefore modify our Dirac operator $\mathcal{D}_t$ as:

$$\mathcal{D}_t \to \mathcal{D}'_t = \tilde{U}\mathcal{D}_t\tilde{U}^\dagger \quad ; \quad \tilde{U} = \mathbb{1} \otimes U \tag{14}$$

appropriately choosing $\hat{U}$ gives:

$$\mathcal{D}'_t = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & \hat{b}^\dagger \\ \hat{b} & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \Lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{15}$$

algebra elements also change in similar fashion to preserve "Ball" condition.

Double Moyal Plane: new Dirac eigen spinors

With this new dirac operator $\mathcal{D}'_t$ we also obtain following new eigen spinors:

$$\left|m, \frac{1}{2}\right\rangle_{\pm} = N_m[u^m_{+\uparrow} + u^m_{-\downarrow} \pm u^m_{-\uparrow}(\sqrt{\kappa m + 1} \mp \sqrt{\kappa m}) \mp u^m_{+\downarrow}(\sqrt{\kappa m + 1} \pm \sqrt{\kappa m})]$$

$$\left|m, -\frac{1}{2}\right\rangle_{\pm} = N_m[u^m_{+\downarrow} + u^m_{-\uparrow} \pm u^m_{+\uparrow}(\sqrt{\kappa m + 1} \pm \sqrt{\kappa m}) \mp u^m_{-\downarrow}(\sqrt{\kappa m + 1} \mp \sqrt{\kappa m})]$$

$$u^m_{\pm\uparrow} = \begin{pmatrix} |m\rangle \\ \pm|m-1\rangle \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \quad u^m_{\pm\downarrow} = \begin{pmatrix} |m\rangle \\ \pm|m-1\rangle \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Double Moyal Plane: new Dirac eigen spinors

- In this new spinorial basis the symmetry is preserved and the longitudinal distance comes same for both copies of Moyal planes.
- The lower bound for transverse ditance comes out to be $d_t = \sqrt{2\theta}$, which can be equated with the distance between two points which is $d_{\mathcal{D}_2} = \frac{1}{|\Lambda|}$. This gives $\kappa = 4$.
- With this fixed value of κ we get the lower bound results:

$$d_l = \frac{2\sqrt{2}|dz|}{|\Lambda|\sqrt{17+\sqrt{65}}} \; ; \; d_t = \frac{1}{|\Lambda|} \quad \textit{and}$$

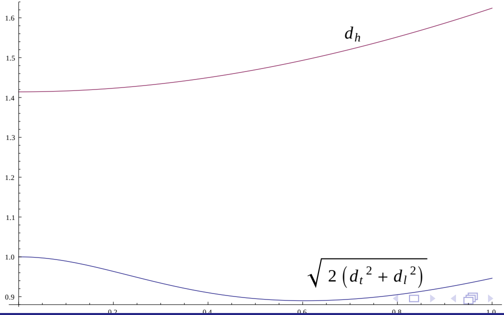
$$d_h = \frac{2\sqrt{2}(1+|dz|^2)}{|\Lambda|\sqrt{4+17|dz|^2+\sqrt{16+136|dz|^2+65|dz|^4}}} \tag{16}$$

Double Moyal Plane: Pythagoras inequality

The Pythagoras inequality for non-unital algebra is the following:

$$d(\Omega_1 = \rho' \otimes \omega_1, \Omega_2 = \rho \otimes \omega_2) \leq \sqrt{2}\sqrt{(d(\rho', \rho))^2 + (d(\omega_1, \omega_2))^2}$$

which for our case will become: $d_h \leq \sqrt{2}\sqrt{d_t^2 + d_l^2}$, which can easily be seen to satisfy the figure below.



What's next?

- To find the upper bound distances for double Moyal plane, and thus compute the most trustworthy distance.
- Verify the Pythagorean inequality for such distances, and also unitise the algebra to be able to verify the main Pythagorean equality.
- Study field dynamics say Electromagnetism in such a non-commutative geometrical setup.

Thank You!