Yang–Mills solutions on Minkowski space via non-compact coset spaces

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Overview

1. Geometry
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2. Lie algebra
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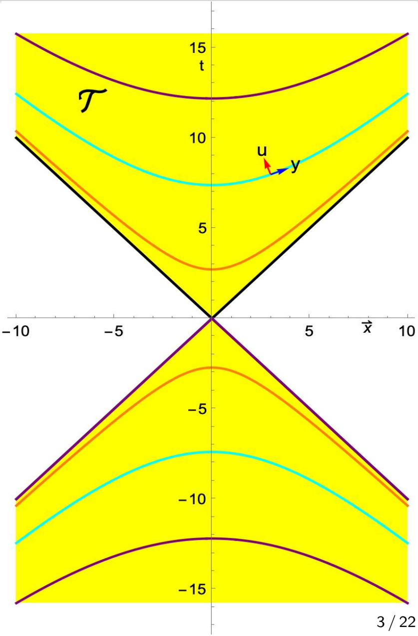
The 2-sheeted hyperboloid H^3_ℓ with radius ℓ can be embedded in Minkowski space $\mathbb{R}^{1,3}$ in the metric $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)_{\mu\nu}$ as

$$\eta_{\mu\nu} Y^\mu Y^\nu = -\ell^2, \ \mu,\nu = 0,1,2,3. \quad (1)$$

We choose to make our coordinates on the hyperbola $Y^\mu \in H^3_\ell$ and inside the lightcone $X^\mu \in \mathcal{T}$ dimensionless using the global length scale ℓ:

$$y^\mu := \frac{Y^\mu}{\ell}, \quad \& \quad x^\mu := \frac{X^\mu}{\ell}. \quad (2)$$



The interior of the lightcone $\mathcal{T}$ can be foliated with unit-hyperboloids $H^3 := H^3_{\ell=1}$ using the following maps:

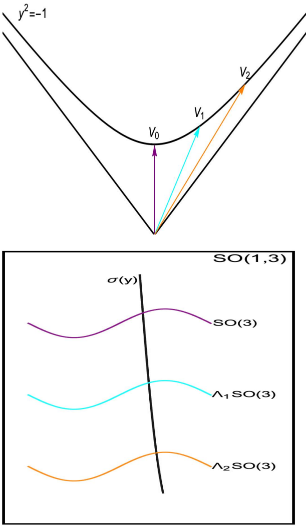
$$\varphi_{\mathcal{T}} : \ \mathbb{R} \times H^3 \to \mathcal{T} \ , \quad (u, y^\mu) \mapsto e^u \, y^\mu =: x^\mu$$
$$\varphi^{-1}_{\mathcal{T}} : \ \mathcal{T} \to \mathbb{R} \times H^3 \ , \quad x^\mu \mapsto \left(\ln \sqrt{-\eta_{\mu\nu} x^\mu x^\nu}, \frac{x^\mu}{\sqrt{-\eta_{\mu\nu} x^\mu x^\nu}} \right) \tag{3}$$

with $e^u = \sqrt{-\eta_{\mu\nu} x^\mu x^\nu}$. Convention: $x^0{=}t, x^1{=}x, x^2{=}y, x^3{=}z$.
The metric on $\mathcal{T}$ becomes conformal to that of a Lorentzian cylinder $\mathbb{R} \times H^3$:

$$\mathrm{d}s^2_{\mathcal{T}} = e^{2u} \left(-\mathrm{d}u^2 + \mathrm{d}s^2_{H^3} \right) \ . \tag{4}$$

Canonical rotation (J_a) and boost (K_a) generators of $SO(1,3)$:

$$K_1 = \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{pmatrix},$$
$$J_1 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0 \end{pmatrix}. \tag{5}$$



- A given timelike vector V_i on the hyperbola can be brought to the vector $V_0 = (1,0,0,0)^T$ via a boost Λ_i.
- The vector V_i is uniquely associated with a coset $\Lambda_i\, SO(3)$ due to the fact that its stabilizer is nothing but $\Lambda_i\, SO(3)\, \Lambda_i^{-1}$.
- The hyperbola H^3 is the curve σ, parameterised by y, that passes through every coset once.
- Alternatively, H^3 is *the* orbit under the adjoint action of the Lorentz group $SO(1,3)$ on the coset space.

The 2-sheeted unit hyperboloid H^3 can be realized as the coset space G/H with $G = SO(1,3)$ and $H = SO(3)$ using the following maps

$$\begin{aligned} \alpha_{\mathcal{T}} &: \; SO(1,3)/SO(3) \to H^3 \;, \quad [\Lambda] \mapsto \Lambda_{0\mu} = y^\mu \\ \alpha_{\mathcal{T}}^{-1} &: \; H^3 \to SO(1,3)/SO(3) \;, \quad y^\mu \mapsto [\Lambda_{\mathcal{T}}] \;, \end{aligned} \tag{6}$$

where

$$\Lambda_{\mathcal{T}} = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta} \\ \gamma\boldsymbol{\beta}^T & \mathbb{1} + (\gamma - 1)\frac{\boldsymbol{\beta}\otimes\boldsymbol{\beta}}{\boldsymbol{\beta}^2} \end{pmatrix} \tag{7}$$

and

$$\beta^i \;=\; \frac{y^i}{y^0} \;, \quad \gamma \;=\; \frac{1}{\sqrt{1 - \vec{\beta}^2}} \;. \tag{8}$$

Notice that the generic expression of the boost $\Lambda_{\mathcal{T}}$ (7) can be obtained by exponentiation with boost generators K_a:

$$\Lambda_{\mathcal{T}} = \exp\left(\eta_a K_a\right) \;; \quad \beta_i = \frac{\eta_i}{\sqrt{\eta^2}} \tanh \sqrt{\eta^2} \;. \tag{9}$$

The exterior of the lightcone $\mathcal{S}$ can be foliated with de Sitter space dS_3 of radius ℓ obeying

$$\eta_{\mu\nu}\, Y^\mu\, Y^\nu \;=\; \ell^2\ . \tag{10}$$

We again work with dimensionless coordinates e.g. $y^\mu := Y^\mu/\ell$. To see the foliation explicitly we note down the following maps

$$\begin{aligned} \varphi_{\mathcal{S}} &:\ \mathbb{R}\times\mathrm{dS}_3 \to \mathcal{S}\ , \quad (u, y^\mu) \mapsto \mathrm{e}^u\, y^\mu =: x^\mu \\ \varphi_{\mathcal{S}}^{-1} &:\ \mathcal{S} \to \mathbb{R}\times\mathrm{dS}_3\ , \quad x^\mu \mapsto \left(\ln\sqrt{\eta_{\mu\nu}x^\mu x^\nu}, \frac{x^\mu}{\sqrt{\eta_{\mu\nu}x^\mu x^\nu}}\right)\ , \end{aligned} \tag{11}$$

where $\mathrm{e}^u = \sqrt{\eta_{\mu\nu}x^\mu x^\nu}$. The metric of $\mathcal{S}$ under this map becomes conformal to a Euclidean cylinder $\mathbb{R}\times\mathrm{dS}_3$:

$$\mathrm{d}s_{\mathcal{S}}^2 = e^{2u}\left(\mathrm{d}u^2 + \mathrm{d}s_{\mathrm{dS}_3}^2\right)\ , \tag{12}$$

where $\mathrm{d}s_{\mathrm{dS}_3}^2$ is the metric on dS_3 induced from (10), and the parameter u is spatial.

For the coset $SO(1,3)/SO(1,2)$ associated with the base vector $(0,0,0,1)$, the splitting (20) is realized by

$$I_a \in \{J_1, J_2, K_3\} \quad \text{and} \quad I_i \in \{K_1, K_2, J_3\}\ . \tag{13}$$

The following map illustrate the equivalence between dS_3 and $SO(1,3)/SO(1,2)$

$$\begin{aligned} \alpha_{\mathcal{S}} &: \ SO(1,3)/SO(1,2) \to \mathrm{dS}_3\ , \quad [\Lambda] \mapsto \Lambda_{0\mu} =: y^\mu \\ \alpha_{\mathcal{S}}^{-1} &: \ \mathrm{dS}_3 \to SO(1,3)/SO(1,2)\ , \quad y^\mu \mapsto [\Lambda_{\mathcal{S}}]\ , \end{aligned} \tag{14}$$

where the representative element $\Lambda_{\mathcal{S}}$ is defined as follows

$$\Lambda_{\mathcal{S}} = \begin{pmatrix} 1+(\gamma-1)\frac{\beta_1^2}{\boldsymbol{\beta}^2} & -(\gamma-1)\frac{\beta_1\beta_2}{\boldsymbol{\beta}^2} & (\gamma-1)\frac{\beta_1\beta_3}{\beta^2} & \beta_1\gamma \\ (\gamma-1)\frac{\beta_1\beta_2}{\boldsymbol{\beta}^2} & 1-(\gamma-1)\frac{\beta_2^2}{\boldsymbol{\beta}^2} & (\gamma-1)\frac{\beta_2\beta_3}{\boldsymbol{\beta}^2} & \beta_2\gamma \\ -(\gamma-1)\frac{\beta_1\beta_3}{\boldsymbol{\beta}^2} & (\gamma-1)\frac{\beta_2\beta_3}{\boldsymbol{\beta}^2} & 1-(\gamma-1)\frac{\beta_3^2}{\boldsymbol{\beta}^2} & -\beta_3\gamma \\ \beta_1\gamma & -\beta_2\gamma & \beta_3\gamma & \gamma \end{pmatrix} \tag{15}$$

with

$$\beta_a = \frac{y^{a-1}}{y^3}\ ,\ \boldsymbol{\beta}^2 = -\eta_{ab}\beta^a\beta^b\ ,\ \gamma = \frac{1}{\sqrt{1-\boldsymbol{\beta}^2}} \tag{16}$$

Notice the presence of the 3-dimensional Minkowski metric $\eta_{ab} = \mathrm{diag}(1,1,-1)_{ab}$ due to the fact that the stabilizer $SO(1,2)$ here is nothing but the isometry group of $\mathbb{R}^{1,2}$.
One can also obtain $\Lambda_{\mathcal{S}}$, analogous to the previous case (9), by exponentiating the coset generators $I_a \in \mathfrak{m}$ with parameters κ_a:

$$\Lambda_{\mathcal{S}} = \exp(\kappa_1 J_1 + \kappa_2 J_2 + \kappa_3 K_3) \tag{17}$$

and employing the following identification

$$\beta_a := \frac{\kappa_a}{\sqrt{\boldsymbol{\kappa}^2}} \tan\sqrt{\boldsymbol{\kappa}^2}\ ;\quad \boldsymbol{\kappa}^2 = \eta^{ab}\kappa_a\kappa_b\ . \tag{18}$$

For reductive coset spaces G/H the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ with structure constants $f_{AB}{}^{C}$:

$$[I_A, I_B] \;=\; f_{AB}{}^{C}\, I_C \quad \text{with} \quad A, B, C \;=\; 1, ..., 6\ , \tag{19}$$

admits the following splitting into a Lie subalgebra $\mathfrak{h}$ and an orthogonal complement $\mathfrak{m}$ such that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$:

$$\mathfrak{g} \;=\; \mathfrak{h} \oplus \mathfrak{m} \;\Longrightarrow\; \{I_A\} \;=\; \{I_i\} \cup \{I_a\} \text{ with } a = 1, 2, 3 \text{ and } i = 4, 5, 6. \tag{20}$$

In effect, we get the following splitting of the Lie algebra

$$[I_i, I_j] \;=\; f_{ij}{}^{k} I_k\ , \quad [I_i, I_a] \;=\; f_{ia}{}^{b} I_b \quad \text{and} \quad [I_a, I_b] \;=\; f_{ab}{}^{i} I_i + f_{ab}{}^{c} I_c\ , \tag{21}$$

where $f_{ab}{}^{c} = 0$ for the symmetric spaces. The Cartan–Killing metric is given by

$$g_{AB} = -\mathrm{tr}_{\mathrm{ad}}(I_A\, I_B) = f_{AD}{}^{C}\, f_{CB}{}^{D}\ . \tag{22}$$

The Maurer–Cartan one-forms are given by

$$g^{-1}\,\mathrm{d}g \;=\; \tilde{e}^A\, I_A \quad \text{for} \quad g \in G\ , \tag{23}$$

which can be pulled back to the coset space G/H using any local section $\sigma :\ G/H \supset U \longrightarrow G$ satisfying $\pi \circ \sigma = \mathrm{Id}_U$ to obtain

$$e^A = \sigma^* \tilde{e}^A\ . \tag{24}$$

These one-forms split into

$$\{e^A\} \;=\; \{e^i\} \cup \{e^a\}\ , \tag{25}$$

where $e^i = e^i_a\, e^a$ are linearly dependent on the three e^a in terms of some real functions e^i_a. These one-forms satisfy the following structure equations consistent with (21):

$$\mathrm{d}e^a + f_{ib}{}^{a}\, e^i \wedge e^b \;=\; 0\ , \quad \mathrm{d}e^i + \tfrac12 f_{jk}{}^{i}\, e^j \wedge e^k + \tfrac12 f_{ab}{}^{i}\, e^a \wedge e^b \;=\; 0\ . \tag{26}$$

For the coset $SO(1,3)/SO(3)$ we have $I_i = J_i$ and $I_a = K_a$ such that

$$f_{ij}{}^{k} = \varepsilon_{i-3\,j-3\,k-3}\ ,\quad f_{ia}{}^{b} = \varepsilon_{i-3\,a\,b}\quad \text{and}\quad f_{ab}{}^{i} = -\varepsilon_{a\,b\,i-3}\ . \tag{27}$$

The Cartan–Killing metric, in this case, splits as follows:

$$g_{ij} = 4\,\delta_{i-3\,j-3}\ ,\quad g_{ab} = -4\,\delta_{ab}\quad \text{and}\quad g_{ia} = 0\ . \tag{28}$$

Moreover, the Maurer–Cartan one-forms $\Lambda_{\mathcal{T}}^{-1}\mathrm{d}\Lambda_{\mathcal{T}} = e^a I_a + e^i I_i$ are given by

$$e^a = \left(\delta^{ab} - \frac{y^a\,y^b}{y^0(1+y^0)}\right)\mathrm{d}y^b\ ,\quad e^i = \varepsilon_{i-3ab}\,\frac{y^a}{1+y^0}\,\mathrm{d}y^b\ . \tag{29}$$

Notice that $e^0 := \mathrm{d}u$ & e^a provides, locally, an orthonormal-frame on the cotangent bundle $T^*(U \subset \mathbb{R}\times H^3)$:

$$\mathrm{d}s^2_{cyl} = -\mathrm{d}u^2 + \mathrm{d}s^2_{H^3} = -e^0\otimes e^0 + \delta_{ab}\,e^a\otimes e^b\ . \tag{30}$$

They are pulled back to Minkowski space $\mathbb{R}^{1,3}$ via $\varphi_{\mathcal{T}}$ (3).

For the Lie algebra (13) of $SO(1,3)/SO(1,2)$ we have

$$f_{ij}{}^{k} = \varepsilon_{i-3\,j-3\,k-3}\,(1-2\,\delta_{k6})\;,\;\; f_{ia}{}^{b} = \varepsilon_{i-3\;a\,b}\,(1-2\,\delta_{a3})\;,\;\; f_{ab}{}^{i} = \varepsilon_{a\,b\,i-3}\;, \tag{31}$$

where the indices for the terms inside the bracket are not summed over. These structure coefficients produce the following Cartan–Killing metric (22)

$$g_{ij} \;=\; 4\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{i-3\,j-3}\;,\;\; g_{ab} \;=\; 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}_{ab}\;,\;\; g_{ia} \;=\; 0\;. \tag{32}$$

The Maurer–Cartan one-forms $\Lambda_{\mathcal{S}}^{-1}\,\mathrm{d}\Lambda_{\mathcal{S}} = e^a\, I_a + e^i\, I_i$ are

$$e^a = \mathrm{d}y^{3-a} - \frac{y^{3-a}}{1+y^3}\,\mathrm{d}y^3\;,\qquad e^i = -\varepsilon_{i-3\;a\,b}\,\frac{y^{3-a}}{1+y^3}\,\mathrm{d}y^{3-b}\;, \tag{33}$$

that give rise to the following metric on the cylinder $\mathbb{R}\times \mathrm{dS}_3$ (12)

$$\tilde{g} \;=\; e^0\otimes e^0 + \eta_{ab}\, e^a\otimes e^b\;. \tag{34}$$

We shall consider our Yang-Mills theory on the Lorentzian cylinder $\mathbb{R} \times G/H$ and the solution obtained therein would be transported back to Minkowski space owing to conformal invariance of the theory in 4D.

For the gauge group G a generic connection one-form $\mathcal{A}$ in "temporal" gauge $\mathcal{A}_0 = 0$ is written as

$$\mathcal{A} = e^i\, I_i + e^a\, X_a \ , \tag{35}$$

which after imposing G-invariance[1] leads to

$$[I_i, X_a] \ = \ f_{ia}^{\ \ b}\, X_b \ . \tag{36}$$

Furthermore, for the symmetric spaces obeying $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ like H^3 and dS_3 the G-invariance makes X_a proportional to I_a:

$$\mathcal{A} = e^i\, I_i + \phi(u) e^a\, I_a \ . \tag{37}$$

[1] D. Kapetanakis & G. Zoupanos, *Phys. Rept.* **219** (1992) 1.

The field strength $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ for the above connection $\mathcal{A}$ takes the following form ($\dot{\phi} := \partial_u \phi$)

$$\mathcal{F} = \dot{\phi}\, I_a\, e^0 \wedge e^a + \frac{1}{2}(\phi^2 - 1)\, f_{ab}{}^{i}\, I_i\, e^a \wedge e^b \ . \tag{38}$$

The Yang–Mills action simplifies to that of a mechanical 'particle' moving in a Newtonian potential $V(\phi)$ for both cases:

$$\begin{aligned} S_{YM} &= -\frac{1}{2e^2} \int_{\mathbb{R}\times H^3/\mathrm{d}S_3} \mathrm{tr}_{\mathrm{ad}}(\mathcal{F} \wedge *\mathcal{F}) \\ &= \frac{12}{e^2} \int_{\mathbb{R}\times H^3/\mathrm{d}S_3} \mathrm{dvol}\, \left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right) \ ; \ V(\phi) = -\frac{1}{2}(\phi^2-1)^2 \ , \end{aligned} \tag{39}$$

which has the following equation of motion

$$\ddot{\phi} \ = \ 2\,\phi\,(\phi^2 - 1) \ . \tag{40}$$

This equation admits a generic solution $\phi_{\epsilon,u_0}(u)$ characterized by its energy $\epsilon = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ and by a shift parameter u_0 in terms of the Jacobi sine function

$$\phi_{\epsilon,u_0}(u) \;=\; f_-(\epsilon)\,\mathrm{sn}\big(f_+(\epsilon)(u-u_0), k\big) \tag{41}$$

with

$$f_\pm(\epsilon) \;=\; \sqrt{1 \pm \sqrt{-2\epsilon}}\,, \quad k^2 \;=\; \frac{f_-(\epsilon)}{f_+(\epsilon)} \tag{42}$$

Special cases, including the “kink”, of $\phi_{\epsilon,u_0}(u)$ are as follows:

$$\phi \;=\; \begin{cases} 0\,, & \epsilon = -\frac{1}{2} \\ \tanh(u-u_0)\,, & \epsilon = 0 \\ \mp 1\,, & \epsilon = 0\,;\; u_0 \to \pm\infty \end{cases} \;. \tag{43}$$

It is a straightforward exercise to pull the orthonormal-frame on $\mathbb{R}\times H^3$ back to $\mathcal{T}$ with the map $\varphi_{\mathcal{T}}$ (3) to get ($r=\sqrt{\vec{x}^2}$)

$$\begin{aligned} e^0 &:= \mathrm{d}u = \frac{t\,\mathrm{d}t - r\,\mathrm{d}r}{t^2-r^2}\,, \\ e^a &= \frac{\mathrm{d}x^a}{\sqrt{t^2-r^2}} - \frac{x^a\left(\sqrt{t^2-r^2}-t\right)\mathrm{d}r}{r(t^2-r^2)} - \frac{x^a\,\mathrm{d}t}{t^2-r^2}\,, \end{aligned} \tag{44}$$

which yields the veirbein components $e^0{=}e^0_\mu\,\mathrm{d}x^\mu$ & $e^a{=}e^a_\mu\,\mathrm{d}x^\mu$. The colour electric $E_i := F_{0i}$ and magnetic $B_i := \frac{1}{2}\varepsilon_{ijk}\,F_{jk}$ fields comes out to be

$$\begin{aligned} E_a &= -\frac{1}{(t^2-r^2)^{3/2}}\left[\varepsilon_{ab}{}^{k-3}\,x^b\,I_k + \left(\frac{x^a\,x^b}{r^2}\left(\sqrt{t^2-r^2}-t\right)+t\,\delta^{ab}\right)I_b\right] \\ B_a &= \frac{1}{(t^2-r^2)^{3/2}}\left[\left(\frac{x^a\,x^{i-3}}{r^2}\left(\sqrt{t^2-r^2}-t\right)+t\,\delta^{a\,i-3}\right)I_i - \varepsilon_{ab}{}^{c}\,x^b\,I_c\right] \end{aligned} \tag{45}$$

The stress-energy tensor on the cylinder $\mathbb{R}\times H^3$ has the form

$$T^{\mathrm{cyl}}_{\mu\nu} \;=\; -\frac{1}{2g^2}\,\mathrm{tr}\Big(\mathcal{F}_{\mu\alpha}\,\mathcal{F}_{\nu\beta}\,\eta^{\alpha\beta} - \tfrac14\eta_{\mu\nu}\mathcal{F}^2\Big)\;;\quad \mathcal{F}^2 \;=\; \mathcal{F}_{\mu\nu}\,\mathcal{F}^{\mu\nu}\;, \tag{46}$$

and computes to

$$T^{\mathrm{cyl}} \;=\; -\frac{6\,\epsilon}{g^2}\,\big(e^0\otimes e^0 + \tfrac13 e^a\otimes e^a\big) \quad\Longrightarrow\quad T^{\mathrm{cyl}\,\mu}_{\;\;\mu} \;=\; \eta^{\mu\nu}\,T^{\mathrm{cyl}}_{\mu\nu} \;=\; 0\,. \tag{47}$$

Pulling T^{cyl} back to the lightcone interior $\mathcal{T}$ using the vierbein components $\{e^0_\mu, e^a_\mu\}$ yields a remarkably simple result:

$$T \;=\; C_{\mathcal{T}}\begin{pmatrix} 3t^2+r^2 & -4tx & -4ty & -4tz \\ -4tx & t^2+4x^2-r^2 & 4xy & 4xz \\ -4ty & 4xy & t^2+4y^2-r^2 & 4yz \\ -4tz & 4xz & 4yz & t^2+4z^2-r^2 \end{pmatrix} \tag{48}$$

with $C_{\mathcal{T}} = -\frac{2\,\epsilon}{g^2(r^2-t^2)^2}$ and vanishing trace, as expected.

Again, the local orthonormal-frame on $\mathbb{R} \times \mathrm{dS}_3$ can be pulled back to the exterior of lightcone $\mathcal{S}$ via the map $\varphi_{\mathcal{S}}$ (11) to obtain

$$
\begin{aligned}
e^0 \;&:=\; \mathrm{d}u = \frac{r\,\mathrm{d}r - t\,\mathrm{d}t}{r^2 - t^2}\ , \\
e^a \;&=\; \frac{\mathrm{d}y^{3-a}}{\sqrt{r^2-t^2}} - \frac{y^{3-a}\,\mathrm{d}y^3}{r^2-t^2+y^3\sqrt{r^2-t^2}} \\
&\qquad - \left(1 - \frac{y^3}{\sqrt{r^2-t^2}+y^3}\right)\frac{y^{3-a}\,e^0}{\sqrt{r^2-t^2}}\ .
\end{aligned}
\tag{49}
$$

This yields the veirbein components

$$
e^0 \;=\; e^0_\mu\,\mathrm{d}x^\mu \quad \text{and} \quad e^a \;=\; e^a_\mu\,\mathrm{d}x^\mu \tag{50}
$$

on the lightcone exterior $\mathcal{S}$.

As before, we compute the stress-energy tensor on the cylinder $\mathbb{R}\times \mathrm{dS}_3$ to get

$$T^{\mathrm{cyl}} \;=\; -\frac{2\,\epsilon}{g^2}\left(e^0\otimes e^0 - e^1\otimes e^1 - 3\,e^2\otimes e^2 + e^3\otimes e^3\right)\;, \qquad (51)$$

which is traceless:

$$T^{\mathrm{cyl}\,\mu}_{\mu} \;=\; \tilde{g}^{\mu\nu}\,T^{\mathrm{cyl}}_{\mu\nu} \;=\; 0\;. \qquad (52)$$

Here again, we use the vierbein components $\{e^0_\mu, e^a_\mu\}$ to pull T^{cyl} back to $\mathcal{S}$:

$$T \;=\; C_{\mathcal{S}}\begin{pmatrix} 3t^2+r^2 & -4tx & -4ty & -4tz \\ -4tx & t^2+4x^2-r^2 & 4xy & 4xz \\ -4ty & 4xy & t^2+4y^2-r^2 & 4yz \\ -4tz & 4xz & 4yz & t^2+4z^2-r^2 \end{pmatrix} \qquad (53)$$

with $C_{\mathcal{S}} = -\frac{2\,\epsilon}{g^2(r^2-t^2)^3}$. This is again traceless, as required.

Summary and outlook

- We have obtained new YM solutions on the interior resp. exterior of lightcone using the Lorentz group $SO(1,3)$ and stabilizer subgroup $SO(3)$ resp. $SO(1,2)$.
- One has the coset $SO(1,3)/ISO(2)$ for the null hypersurface, but there is no foliation and hence no dynamics.
- Yang–Mills dynamics can be explored on other related/unrelated coset spaces like $SO(2,2)/SO(1,2)$ and $G_2/SU(3)$ in future works.

Thank You!

Questions?