ANGULAR MOMENTUM AND SIMPLE HARMONIC OSCILLATOR PROBLEMS IN 2D NON-COMMUTATIVE MOYAL PLANE

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- Quantization of space time is a way out and can also help gain insight into the Plank scale nature of spacetime.
- Low energy regime of String theory also gives rise to such non-commuting spaces.
- has relevance in certain condensed matter phenomena like the quantum Hall effect and topological insulators.

Simplest "toy" model: Moyal plane

$$[\hat{x}_\alpha, \hat{x}_\beta] = i\theta_{\alpha\beta} = i\theta\epsilon_{\alpha\beta} \quad (\alpha, \beta \in \{1, 2\}) \tag{1}$$

which augments the usual Heisenberg algebra:

$$[\hat{x}_\alpha, \hat{p}_\beta] = i\hbar\delta_{\alpha\beta} \;\; ; \;\; [\hat{p}_\alpha, \hat{p}_\beta] = 0 \tag{2}$$

This algebra renders the configuration space being "fuzzy" due to the uncertainty relation:

$$(\Delta x_1)(\Delta x_2) \geq \frac{\theta}{2} \tag{3}$$

One can at best talk about the so called "coherent" states $|z)$ which saturates the above inequality.

Simplest "toy" model: Moyal plane

One can naturally associate a Hilbert space $\mathcal{H}_c$ to the coordiate algebra (1), which is isomorphic to the Hilbert space of 1D qunatum mechanical SHO.

$$\mathcal{H}_c = \text{span}\{|n\rangle = \frac{1}{\sqrt{n!}}(\hat{b}^\dagger)^n |0\rangle\}_{n=0}^{\infty} \tag{4}$$

where the annihilation operator $\hat{b}$ is given by:

$$\hat{b} = \frac{1}{\sqrt{2\theta}}(\hat{x}_1 + i\hat{x}_2) \tag{5}$$

satisfying $[\hat{b}, \hat{b}^\dagger] = 1$ and $\hat{b}\,|0\rangle = 0$. This Hilbert space replaces the 2D configuration space of commutative quantum mechanics.

Simplest "toy" model: Moyal plane

The quantum Hilbert space $\mathcal{H}_q$ then comes naturally as the set of bounded trace class operators on $\mathcal{H}_c$ as:

$$\mathcal{H}_q = \{|\psi), Tr_c(\psi^\dagger \psi) < \infty\}. \tag{6}$$

Also the associated inner product (and thus norm) can easily be defined as:

$$(\phi|\psi) = Tr_c(\phi^\dagger \psi) \tag{7}$$

Now we look for the unitary representation of $\hat{x}_\alpha$ (which unlike in the usual QM are not c-numbers anymore) and their conjugate momenta. Calling them $\hat{X}_i$ and $\hat{P}_i$ one can verify that following representation does the job:

$$\hat{X}_i \psi = \hat{x}_i \psi, \quad \hat{P}_i \psi = \frac{1}{\theta}\epsilon_{ij}[\hat{x}_j, \psi] = \frac{1}{\theta}\epsilon_{ij}\left(\hat{X}_j^L - \hat{X}_j^R\right)\psi, \tag{8}$$

Simplest "toy" model: Moyal plane

Here the left/right action (for all $\psi \in \mathcal{H}_q$) is defined as

$$\hat{X}_i^L \psi \equiv \hat{X}_i \psi = \hat{x}_i \psi \quad ; \quad \hat{X}_i^R \psi = \psi \hat{x}_i \tag{9}$$

Such that these operators $\hat{X}_i$ and $\hat{P}_i$ satisfies the non-commutative Heisengberg algebra

$$[\hat{X}_i^L, \hat{X}_j^L] \equiv [\hat{X}_i, \hat{X}_j] = i\theta\epsilon_{ij} \ ; \ [\hat{X}_i, \hat{P}_j] = i\delta_{ij} \ ; \ [\hat{P}_i, \hat{P}_j] = 0 \tag{10}$$

It's also easy to see that

$$[\hat{X}_i^R, \hat{X}_j^R] = -i\theta\epsilon_{ij} \ ; \ [\hat{X}_i^L, \hat{X}_j^R] = 0 \tag{11}$$

Angular Momentum: Schwinger's way

Using creation ($\hat{a}^\dagger_\alpha$) and annihilation ($\hat{a}_\alpha$) operators of a pair of independent harmonic oscillators satisfying:

$$[\hat{a}_\alpha, \hat{a}_\beta] = 0 = [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta] \text{ and } [\hat{a}_\alpha, \hat{a}^\dagger_\beta] = \delta_{\alpha\beta} \quad \forall \alpha, \beta = 1, 2 \quad (12)$$

One can obtain general angular momentum operators using Pauli matrices as

$$\hat{\vec{J}} = \frac{1}{2}\hat{a}^\dagger_\alpha \{\vec{\sigma}\}_{\alpha\beta}\hat{a}_\beta \ ; \ [\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k, \quad (13)$$

In commutative case we can study decoupled 1D SHO as $|m\rangle\langle n| \in \mathcal{H}_c \otimes \mathcal{H}_c$ with creation operators $\hat{a}^\dagger_1$, $\hat{a}^\dagger_2$ as $(\hat{a}^\dagger \otimes 1)$ and $(1 \otimes \hat{a}^\dagger)$ respectively.

Angular Momentum: Commutative case

With above Schwinger's prescription we can easily obtain following SU(2) generators

$$\hat{J}_1 = \frac{1}{2}\left(\hat{a}_2^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2\right), \ \hat{J}_2 = \frac{i}{2}\left(\hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2\right), \ \hat{J}_3 = \frac{1}{2}\left(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2\right)$$

in which $\hat{J}_3$ and the Casimir $\hat{\vec{J}}^2$ operators satisfy following eigen-value equations:

$$\hat{J}_3\left(|m\rangle \otimes |n\rangle\right) = j_3\left(|m\rangle \otimes |n\rangle\right), \quad \text{where } j_3 = \frac{1}{2}(m-n)$$

$$\hat{\vec{J}}^2\left(|m\rangle \otimes |n\rangle\right) = j(j+1)\left(|m\rangle \otimes |n\rangle\right), \quad \text{where } \ j = \frac{1}{2}(m+n)$$

Angular Momentum: Commutative case

Now considering following Hamiltonian

$$\hat{H} = \frac{1}{2}\mu\omega^2\left(\hat{X}_1^2 + \hat{X}_2^2\right) + \frac{1}{2\mu}\left(\hat{P}_1^2 + \hat{P}_2^2\right) = \frac{\omega}{2}\left(\frac{\hat{\vec{P}}^2}{\mu\omega} + \mu\omega\hat{\vec{X}}^2\right)$$

which under following canonical transformation

$$\hat{P}_\alpha \to \hat{p}_\alpha = \frac{\hat{P}_\alpha}{\sqrt{\mu\omega}} \quad \text{and} \quad \hat{X}_\alpha \to \hat{x}_\alpha = \sqrt{\mu\omega}\hat{X}_\alpha \qquad (14)$$

becomes $\hat{H} = \frac{\omega}{2}\left(\hat{x}_1^2 + \hat{x}_2^2 + \hat{p}_1^2 + \hat{p}_2^2\right)$ which seems to enjoy full $SO(4)$ symmetry in 4D phase-space. However it's only the subgroup SU(2) $\subset$ SO(4) $= SU(2) \otimes SU(2)$ is the relevant symmetry group.

Angular Momentum: Commutative case

Now using the previous form of generators $\hat{J}_i$ in terms of ladder operators and following relations

$$\hat{x}_\alpha = \frac{1}{\sqrt{2}}\left(\hat{a}_\alpha + \hat{a}_\alpha^\dagger\right); \quad and, \quad \hat{p}_\alpha = \frac{i}{\sqrt{2}}\left(\hat{a}_\alpha^\dagger - \hat{a}_\alpha\right); \qquad (15)$$

one can easily obtain following commutation relations

$$[\hat{J}_3, \hat{x}_\alpha] = \frac{i}{2}(\delta_{\alpha 2}\hat{p}_2 - \delta_{\alpha 1}\hat{p}_1) \text{ and } [\hat{J}_3, \hat{p}_\alpha] = \frac{i}{2}(\delta_{\alpha 1}\hat{x}_1 - \delta_{\alpha 2}\hat{x}_2)$$

$$[\hat{J}_1, \hat{x}_\alpha] = \frac{-i}{2}(\delta_{\alpha 1}\hat{p}_2 + \delta_{\alpha 2}\hat{p}_1) \text{ and } [\hat{J}_1, \hat{p}_\alpha] = \frac{i}{2}(\delta_{\alpha 1}\hat{x}_2 + \delta_{\alpha 2}\hat{x}_1)$$

$$[\hat{J}_2, \hat{x}_\alpha] = \frac{i}{2}\epsilon_{\alpha\beta}\hat{x}_\beta \text{ and } [\hat{J}_2, \hat{p}_\alpha] = \frac{i}{2}\epsilon_{\alpha\beta}\hat{p}_\alpha; \quad \text{with} \quad \alpha, \beta = 1, 2.$$

Angular Momentum: Commutative case

Above commutation relation clearly shows that $\hat{J}'s$ generate simultaneous SO(2) rotations in two orthogonal planes like $\hat{J}_3$ generates simultaneous SO(2) rotation in x_1p_1 and x_2p_2 planes and so on.
Now any general state in $\mathcal{H}_q$ can be written as

$$\big|\Psi\big) = \sum_{m,n} C_{mn}|m\rangle\langle n| \in \mathcal{H}_q \tag{16}$$

Thus $\mathcal{H}_q$ can be identified with $\mathcal{H}_c \otimes \tilde{\mathcal{H}}_c$, where $\tilde{\mathcal{H}}_c$ is the dual of $\mathcal{H}_c$. Since, there is a one-to-one map between the basis $|m\rangle \otimes |n\rangle$ and $|m\rangle \otimes \langle n|$, the Hilbert spaces, span$\{|m\rangle \otimes |n\rangle\}$=$\mathcal{H}_c \otimes \mathcal{H}_c$ and $\mathcal{H}_q$ are isomorphic.

Angular Momentum: non-commutative case

Now replacing $\hat{a}_1$ with $\hat{B}_L$ and $\hat{a}_2^\dagger$(and not $\hat{a}_2$) with $\hat{B}_R$ in the commutative case where operators $\hat{B}_L = \hat{b} \otimes 1$ and $\hat{B}_R = 1 \otimes \hat{b}_R$ belong to $\mathcal{H}_q$ we obtain

$$\hat{J}_1 = \frac{1}{2}\left(\hat{B}_R\hat{B}_L + \hat{B}_L^\ddagger\hat{B}_R^\ddagger\right), \quad \hat{J}_2 = \frac{i}{2}\left(\hat{B}_R\hat{B}_L - \hat{B}_L^\ddagger\hat{B}_R^\ddagger\right) \quad \text{and}$$

$$\hat{J}_3 = \frac{1}{2}\left(\hat{B}_L^\ddagger\hat{B}_L - \hat{B}_R\hat{B}_R^\ddagger\right) \tag{17}$$

It's an easy job to check that the $\hat{J}_3$ and Casimir operator $\hat{J}^2$ satisfies the same eigen value equation as in the commutative case. We can use the definition $\hat{B}_{L/R} = \frac{1}{\sqrt{2\theta}}(\hat{X}_1^{L/R} + i\hat{X}_2^{L/R})$ and its Hermitian conjugate, and the adjoint action of momenta we get

Angular Momentum: non-commutative case

$$\hat{X}_1^L = \sqrt{\frac{\theta}{2}}(\hat{B}_L + \hat{B}_L^\ddagger) \ , \ \hat{X}_2^L = i\sqrt{\frac{\theta}{2}}(\hat{B}_L^\ddagger - \hat{B}_L)$$

$$\hat{P}_1 = \frac{i}{\sqrt{2\theta}}\left(\hat{B}_L^\ddagger - \hat{B}_L - \hat{B}_R^\ddagger + \hat{B}_R\right)$$

$$\hat{P}_2 = \frac{1}{\sqrt{2\theta}}\left(\hat{B}_R^\ddagger + \hat{B}_R - \hat{B}_L^\ddagger - \hat{B}_L\right)$$

Further, the commuting coordinates introduced as

$$\hat{X}_i^c = \frac{1}{2}\left(\hat{X}_i^L + \hat{X}_i^R\right) = \hat{X}_i + \frac{\theta}{2}\epsilon_{ij}\hat{P}_j$$

satisfying $[\hat{X}_i^c, \hat{X}_j^c] = 0$, can be expressed like-wise as:

Angular Momentum: non-commutative case

$$\hat{X}_1^c = \frac{1}{2}\sqrt{\frac{\theta}{2}}\left(\hat{B}_R + \hat{B}_L + \hat{B}_L^{\ddagger} + \hat{B}_R^{\ddagger}\right)$$

$$\hat{X}_2^c = \frac{i}{2}\sqrt{\frac{\theta}{2}}\left(\hat{B}_L^{\ddagger} - \hat{B}_L + \hat{B}_R^{\ddagger} - \hat{B}_R\right)$$

We now perform canonical transformation similar to the commutative case

$$\hat{X}_i^c \rightarrow \hat{x}_i^c = \frac{\hat{X}_i^c}{\sqrt{\theta}} \text{ and } \hat{P}_i \rightarrow \hat{p}_i = \sqrt{\theta}\hat{P}_i \quad \forall\, i = 1, 2$$

We then calculate following commutation relation

$$[\hat{x}_i^c, \hat{J}_1] = \frac{i}{2}\left(\delta_{i1}\hat{p}_1' - \delta_{i2}\hat{p}_2'\right) \text{ and } [\hat{p}_i', \hat{J}_1] = \frac{i}{2}\left(\delta_{i2}\hat{x}_2 - \delta_{i1}\hat{x}_1\right)$$

Angular Momentum: non-commutative case

$$[\hat{x}_i^c, \hat{J}_2] = -\frac{i}{2}\left(\delta_{i1}\hat{p}_2' + \delta_{i2}\hat{p}_1'\right) \text{ and } [\hat{p}_i', \hat{J}_2] = \frac{i}{2}\left(\delta_{i1}\hat{x}_2 + \delta_{i2}\hat{x}_1\right)$$

$$[\hat{x}_i^c, \hat{J}_3] = \frac{i}{2}\epsilon_{ij}\hat{x}_j \text{ and } [\hat{p}_i', \hat{J}_3] = \frac{i}{2}\epsilon_{ij}\hat{p}_j'$$

where $\hat{p}_i' = \frac{\hat{p}_i}{2}$. This shows that the Angular momentum operators generates simultaneous SO(2) rotations in orthogonal planes (not the phase-space!) here as well but the roles of these operators are different then their commutative counterpart

$$\hat{J}_1^{NC} \leftrightarrow \hat{J}_3^{C}, \quad \hat{J}_2^{NC} \leftrightarrow \hat{J}_1^{C} \text{ and } \hat{J}_3^{NC} \leftrightarrow \hat{J}_2^{C}$$

SHO: commutative case

Here the Hamiltonian is given by

$$\hat{H}_I = \frac{1}{2}\mu\omega^2 \hat{\vec{X}}^2 + \frac{1}{2\mu}\hat{\vec{P}}^2 \tag{18}$$

which with the substitution in terms of ladder operators using

$$\hat{\vec{X}}^2 = \frac{1}{2\mu\omega}\left(\hat{a}_1\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_1^\dagger + \hat{a}_2\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_2^\dagger\right) + \frac{1}{\mu\omega}\left(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 + 1\right)$$

$$\hat{\vec{P}}^2 = -\frac{\mu\omega}{2}\left(\hat{a}_1\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_1^\dagger + \hat{a}_2\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_2^\dagger\right) + \mu\omega\left(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 + 1\right),$$

so that we get SU(2) invariant Hamiltonian

$$\hat{H}_I = \omega\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right),$$

whose spectrum is just $\hat{H}_I |m\rangle \otimes |n\rangle = \omega(2j+1)|m\rangle \otimes |n\rangle$ where $j = \frac{1}{2}(m+n)$ as expected.

SHO: non-commutative case

Let's first study the unphysical SHO (involving $\hat{X}_i^c$). The Hamiltonian is

$$\hat{H}_2 = \frac{1}{2\mu}\hat{\vec{P}}^2 + \frac{1}{2}\mu\omega^2(\hat{\vec{X}}^c)^2 \tag{19}$$

We might naively expect this to be similar to commutative case, which is not completely true. Let's construct the ladder operators here

$$\hat{C}_i^\dagger = \frac{1}{\sqrt{2\mu\omega}}\left(\mu\omega\hat{X}_i^c - i\hat{P}_i\right), \;\; \hat{C}_i = \frac{1}{\sqrt{2\mu\omega}}\left(\mu\omega\hat{X}_i^c + i\hat{P}_i\right) \;\; \forall \;\; i = 1, 2 \tag{20}$$

The corresponding ground state $|\Omega) \in \mathcal{H}_q$ is then defined as $\hat{C}_i|\Omega) = 0$. Note that there is another "Vacuum" state $|0\rangle\langle 0|$ satisfying $\hat{B}_L|0\rangle\langle 0| = 0 = \hat{B}_R^\ddagger|0\rangle\langle 0|$.

SHO: non-commutative case

These two matches only under a special choice of parameters μ and ω, which we call as a critical point:

$$\mu_0 = \frac{\omega_0}{2} = \frac{1}{\sqrt{\theta}} \tag{21}$$

At this critical point our Hamiltonian $\hat{H}_2$ becomes

$$\hat{H}_2 = \omega_0 \left(\frac{1}{\theta} (\hat{\vec{X}}^c)^2 + \frac{\theta}{4} \hat{\vec{P}}^2 \right) = \omega_0 \left(\hat{B}_L^\ddagger \hat{B}_L + \hat{B}_R \hat{B}_R^\ddagger + 1 \right),$$

which reproduces the spectrum of commutative case i.e. $E = \omega_0(2j + 1)$.

SHO: non-commutative case

Now for arbitrary values of parameters μ and ω Hamiltonian takes the form

$$\hat{H}_2 = \alpha\left(\hat{B}_L^{\ddagger}\hat{B}_L + \hat{B}_R^{\ddagger}\hat{B}_R\right) + \beta\left(\hat{B}_L^{\ddagger}\hat{B}_R + \hat{B}_R^{\ddagger}\hat{B}_L\right) \tag{22}$$

where

$$\alpha = \frac{\mu\omega^2\theta}{4} + \frac{1}{\mu\theta} \quad \text{and} \quad \beta = \frac{\mu\omega^2\theta}{4} - \frac{1}{\mu\theta} \tag{23}$$

To diagonalize the Hamiltonian we perform following Bogoliubov transformation

$$\begin{pmatrix} \hat{B}'_L \\ \hat{B}'_R \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} \hat{B}_L \\ \hat{B}_R \end{pmatrix} \tag{24}$$

This ensures $[\hat{B}'_L, \hat{B}'^{\ddagger}_L] = -[\hat{B}'_R, \hat{B}'^{\ddagger}_R] = 1$. So that we get

SHO: non-commutative case

$$\hat{H}_2 = \left[\alpha\left(\cosh^2\phi + \sinh^2\phi\right) - 2\beta\sinh\phi\cosh\phi\right]\left(\hat{B}_L'^{\ddagger}\hat{B}_L' + \hat{B}_R'\hat{B}_R'^{\ddagger}\right)$$
$$+ \left[\beta\left(\cosh^2\phi + \sinh^2\phi\right) - 2\alpha\sinh\phi\cosh\phi\right]\left(\hat{B}_L'^{\ddagger}\hat{B}_R' + \hat{B}_R'^{\ddagger}\hat{B}_L'\right)$$
$$+ 2\alpha\sinh^2\phi - 2\beta\sinh\phi\cosh\phi + \alpha$$

Setting the coefficient of the off-diagonal term to zero we get

$$\coth\phi + \tanh\phi = \frac{2\alpha}{\beta} \tag{25}$$

which gives two roots with the smaller one being $\tanh\phi = \frac{1}{\beta}(\alpha - \omega)$. Also the Hamiltonian becomes (with this solution)

$$\hat{H}_2 = \omega\left(B_L'^{\ddagger}B_L' + B_R'B_R'^{\ddagger} + 1\right) \tag{26}$$

SHO: non-commutative case

This again reproduces the spectrum of commutative case with arbitrary parameter ω (rather than ω_0). Also note that the Bogoliubov transformation used above is actually equivalent to following canonical transformation

$$\hat{X}_i'^c = e^{\phi}\hat{X}_i^c \; ; \quad \hat{P}_i' = e^{-\phi}\hat{P}_i \tag{27}$$

where $e^{\phi} = \sqrt{\frac{\mu\omega\theta}{2}}$ for the particular solution we have choosen. This transformation is in turn equivalent to following unitary transformation

$$\hat{X'}_i^c = e^{\phi}\hat{X}_i^c = e^{-\frac{i}{2}\phi\hat{D}}\hat{X}_i^c e^{\frac{i}{2}\phi\hat{D}}, \tag{28}$$

where $\hat{D} = \frac{1}{2}(\hat{X}_i^c\hat{P}_i + \hat{P}_i\hat{X}_i^c) = i(\hat{B}_L^\ddagger\hat{B}_R - \hat{B}_L\hat{B}_R^\ddagger)$ is the dilatation operator.

SHO: non-commutative case

Now as for the physical Hamiltonian for SHO in non-commutative case we have

$$\hat{H}_3 = \frac{1}{2\mu}\hat{\vec{P}}^2 + \frac{1}{2}\mu\omega^2(\hat{\vec{X}})^2 \tag{29}$$

Re-writing $\hat{H}_3$ in terms of $\hat{X}_i^c$, by eliminating $\hat{X}_i$ we obtain

$$\hat{H}_3 = \frac{1}{2\mu}\hat{\vec{P}}^2 + \frac{1}{2}\mu\omega^2\left[(\hat{\vec{X}}^c)^2 + \frac{\theta^2}{4}\hat{\vec{P}}^2 + 2\theta\hat{J}_3\right]. \tag{30}$$

which after re-organizing can be written as

$$\hat{H}_3 = \frac{1}{2\mu'}\hat{\vec{P}}^2 + \frac{1}{2}\mu'\omega'^2(\hat{\vec{X}}^c)^2 + \mu\theta\omega^2\hat{J}_3, \tag{31}$$

SHO: non-commutative case

where the renormalized paramters μ' and ω', satisfying $\mu\omega^2 = \mu'{\omega'}^2$, are given by:

$$\frac{1}{\mu'} = \frac{1}{\mu} + \frac{\mu\omega^2\theta^2}{4} \quad \text{and} \quad \omega'^2 = \omega^2\left(1 + \frac{\mu^2\omega^2\theta^2}{4}\right)$$

Now the extra Zeeman term in the Hamiltonian breaks the SU(2) symmetry to U(1) symmetry. The spectrum can easily be read off using previous excercise unphysical SHO

$$E(j, j_3) = \omega'(2j+1) + \theta\mu'{\omega'}^2 j_3 \tag{32}$$

This clearly suggests that these renormalised parameters ω' and μ' rather than their 'bare' counterparts ω and μ, which are the observable quantities of the theory.

Missed out stuff and current interest

- time-reversal symmetry breaking due to Zeeman term
- analogy with squeezed coherent state (Eur. Phys. J. Plus (2015) 130: 120)
- I'm currently involved in studying the geometry of such non-commutative spaces using the Connes spectral triple formalism.

Thank You!