Geodesic flows on a black-hole background

Kaushlendra Kumar and Shahn Majid

Quantum Algebra seminar, QMUL

9 Dec 2025

Geodesic flow equations from NCG

The geodesic velocity equation (dot is for $\frac{\mathrm{d}}{\mathrm{d}s}$):

$$\dot{X} + \nabla_X X = 0$$

Density flow equation:

$$\dot{\rho} + X(\rho) + \rho\,\mathrm{div}(X) = 0$$

Amplitude ($\rho = |\psi|^2$) flow equation:

$$\dot{\psi} + X(\psi) + \frac{1}{2}\psi\,\mathrm{div}(X) = 0$$

For $\psi = \sqrt{\rho}e^{\mathrm{i}\theta}$:

$$\dot{\theta} + X(\theta) = 0$$

Further equations

Convective derivative (relativistic fluid mechanics):

$$\frac{D}{Ds}|_X f = \dot{f} + X(f), \quad \frac{D}{Ds}|_X Y = \frac{\mathrm{d}Y}{\mathrm{d}s} + \nabla_X Y$$

Derivatives of norm and divergence:

$$\frac{D}{Ds}|_X |X|^2 = 0, \quad \frac{D}{Ds}|_X (\mathrm{div}(X)) = -(\nabla_\mu X^\nu)(\nabla_\nu X^\mu) - X^\mu X^\nu R_{\mu\nu}$$

Consequence for initial velocity field $X(0)$:

1. unit-speed: $|X|^2 = 2g_{UV}X^U X^V = -1$ remains invariant
2. $X(0)$ as infinitesimal diffeomorphism, i.e. $\rho(s)(x) = \rho(x - sX(x)) \implies \dot{\rho}(s)(x) = -X(\rho(s))$ at $s = 0$, so that $\mathrm{div}(X(0)) = 0$. This condition changes with evolution.

Kruskal-Szekeres coordinates

Schwarzschild metric for Black Hole:

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{r_s}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

With $r_s = 2M$, $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. For $r > r_s$:

$$T = \left(\frac{r}{r_s} - 1\right)^{1/2} e^{r/(2r_s)} \sinh\left(\frac{t}{2r_s}\right)$$

$$X = \left(\frac{r}{r_s} - 1\right)^{1/2} e^{r/(2r_s)} \cosh\left(\frac{t}{2r_s}\right)$$

For $0 < r < r_s$:

$$T = \left(1 - \frac{r}{r_s}\right)^{1/2} e^{r/(2r_s)} \cosh\left(\frac{t}{2r_s}\right)$$

$$X = \left(1 - \frac{r}{r_s}\right)^{1/2} e^{r/(2r_s)} \sinh\left(\frac{t}{2r_s}\right)$$

Null coordinates U, V

$U = T - X$, $V = T + X$. Expressions in each region provided. Relations:

$$UV = -\left(\frac{r}{r_s} - 1\right) e^{r/r_s}, \quad \frac{U}{V} = -\exp\left(-\frac{t}{r_s}\right)$$

Here

$$r = r_s \left(1 + W_0\left(\frac{-UV}{e}\right)\right)$$

The metric becomes

$$\mathrm{d}s^2 = \frac{4r_s^3}{r} e^{-r/r_s}(-\mathrm{d}U\mathrm{d}V) + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

Relevant Christoffel symbols are:

$$\Gamma^V_{VV} = \frac{r_s^2}{r^2}(1 + \frac{r}{r_s})Ue^{-r/r_s}, \quad \Gamma^U_{UU} = \frac{r_s^2}{r^2}(1 + \frac{r}{r_s})Ve^{-r/r_s}$$

Kruskal diagram

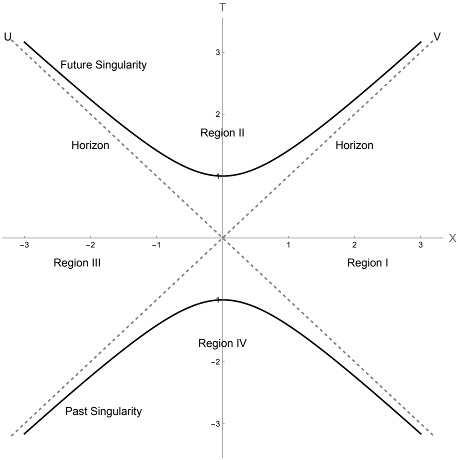


Figure: Kruskal-Szekeres diagram illustrating the four regions of spacetime.

Geodesic bunch as flow of dust in U, V

We work in units of $r_s = 1$. The geodesic equation

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}s^2} + \Gamma^\alpha_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0$$

for radial motion ($\theta = \frac{\pi}{2}$, $\phi = 0$) in (U, V) coordinates with are

$$\frac{\mathrm{d}X^U}{\mathrm{d}s} + \left(\frac{1}{r(U,V)^2} - 1 \right) \frac{(X^U)^2}{U} = 0, \quad \frac{\mathrm{d}X^V}{\mathrm{d}s} + \left(\frac{1}{r(U,V)^2} - 1 \right) \frac{(X^V)^2}{V} = 0.$$

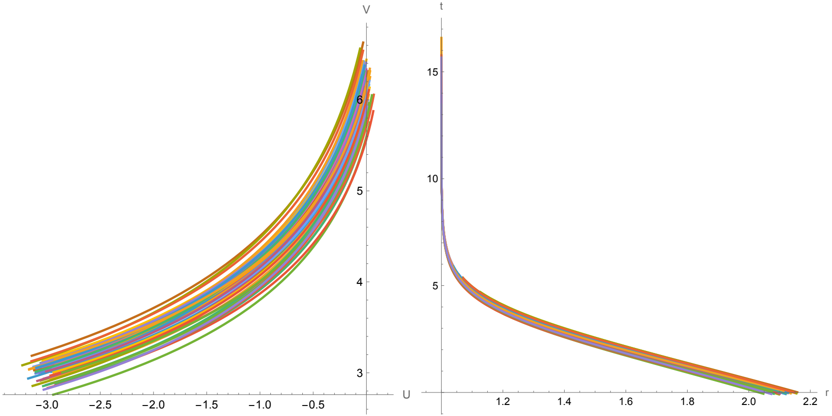
We model a flow of dust as a bunch of nearby geodesics, whose initial positions are sampled from Gaussian ($\sigma = 0.1$):

$$\rho(0)(U, V) := \frac{1}{\sqrt{-g} Z} \exp^{-\frac{(U+3)^2 + (V-3)^2}{2\sigma^2}}.$$

Here, Z ensures that $\int \sqrt{-g} \mathrm{d}U \mathrm{d}V \rho(0)(U, V) = 1$. Moreover, their initial velocity comes from a chosen $X(0)$ with $X^V > 0$, $X^U > 0$ (future directed), i.e.

$$\dot{U}_i(0) = X^U(U_i(0), V_i(0)), \quad \dot{V}_i(0) = X^V(U_i(0), V_i(0)).$$

Geodesic Bunch Plots



(a) Kruskal-Szekeres (b) Schwarzschild

Figure: Parametric plots of geodesics for 50 particles.

Statistical density ρ_{stat} and velocity X_{stat}

We extract statistical density profile from above geodesic bunch at any given s as a Gaussian interpolation:

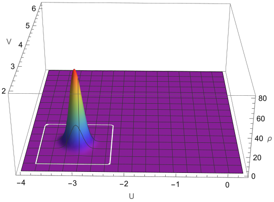
$$\rho_{\rm stat}(s)(U,V) = \frac{1}{\sqrt{-g}Z_s}\sum_{i=1}^{N} \exp(-\frac{(U-U_i(s))^2+(V-V_i(s))^2}{\sigma^2})$$

Similarly, we have (rather crude) statistical X as Gaussian interpolation:

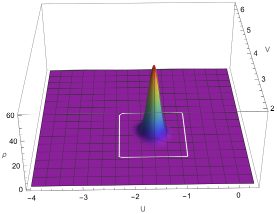
$$X^U_{stat}(s) = \frac{\sum\limits_{i=1}^{N} \rho_{stat}(s)(U_i(s),V_i(s))X^U_i(s)e^{-\frac{(U-U_i(s))^2+(V-V_i(s))^2}{2\sigma^2}}}{\sum\limits_{i=1}^{N} \rho_{stat}(s)(U_i(s),V_i(s))e^{-\frac{(U-U_i(s))^2+(V-V_i(s))^2}{2\sigma^2}}},$$

with $X^V_{stat}(s)$ given in same way.

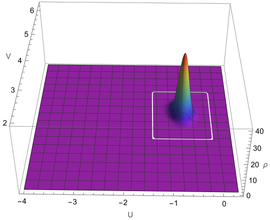
Evolution of ρ_{stat}



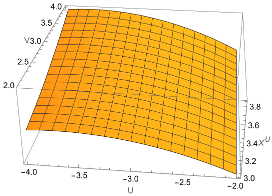
(a) $s = 0$



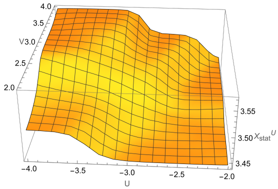
(b) $s = 0.5$



(c) $s = 1$



(d) $X^U(0)$



(e) $X^U_{stat}(0)$

Figure: Evolution of ρ_{stat}, $X^U(0)$ and $X^U_{stat}(0)$ for $N = 50$.

Geodesic & density flow equations in U, V

Next, we evolve the initial data $(\rho(0), X(0))$ used in geodesic bunch case through flow equations:

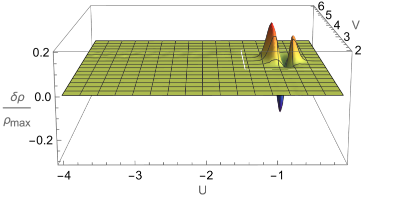
$$\frac{\mathrm{d}X^U}{\mathrm{d}s} + (\partial_U X^U)X^U + (\partial_V X^U)X^V + \left(\frac{1}{r(U,V)^2} - 1\right)\frac{(X^U)^2}{U} = 0,$$

$$\frac{\mathrm{d}X^V}{\mathrm{d}s} + (\partial_U X^V)X^U + (\partial_V X^V)X^V + \left(\frac{1}{r(U,V)^2} - 1\right)\frac{(X^V)^2}{V} = 0,$$

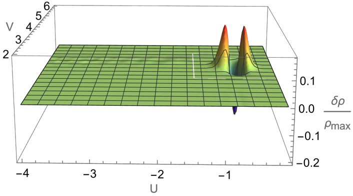
$$\dot{\rho} + \partial_U(\rho X^U) + \partial_V(\rho X^V) + \left(\frac{1}{r(U,V)^2} - 1\right)\left(\frac{X^U}{U} + \frac{X^V}{V}\right)\rho = 0.$$

Note that unit speed: $X^V = \frac{1}{2\sqrt{-g}X^U}$ makes second one redundant. We chose BCs at $U = -4$ such that $\rho(s)(-4, V) = 0$ and $X(s) = X(0)$.

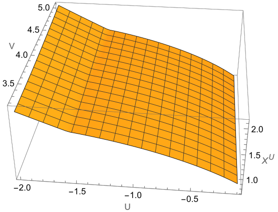
Comparison of evolved densities



(a) $N = 50$



(b) $N = 500$



(c) $X^U(1)$

Figure: 3d plots of difference $\rho_{stat}(1) - \rho(1)$ as fraction of ρ_{max} and evolved

Choosing initial velocity field $X(0)$

Divergence free $\mathrm{div}(X) = 0$ combined with unit speed condition reads

$$\frac{4(X^U)^2}{UV}\Big((1-r^2)VX^U + r^2 UV\partial_U X^U\Big) = \frac{re^r}{UV}\Big((U-V)(r^2-1)X^U + r^2 UV\partial_V X^U$$

We solve this using Dirichlet BC on $(-4, V)$ and Neumann BC on $(U, 7)$ or vice versa. Chosing the boundary values:

1. Covariant constant $\nabla_{\partial_V} X = 0$ on $(-4, V)$ or $\nabla_{\partial_U} X = 0$ on $(U, 7)$,

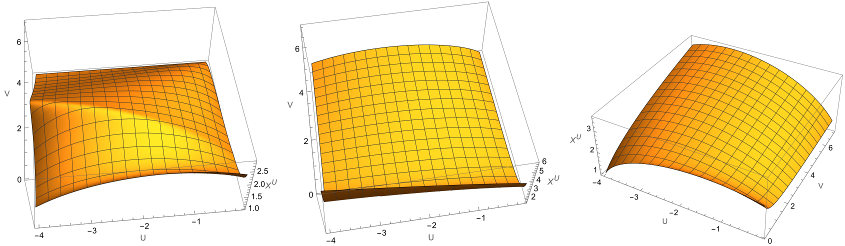
$$X^U_{const\ U_0}(U_0, V) = \frac{c_1}{2}, \quad X^V_{const\ U_0}(U_0, V) = \frac{1}{2c_1} r(U_0, V) e^{r(U_0, V)}$$

$$X^U_{const\ V_0}(U, V_0) = \frac{c_2}{2} r(U, V_0) e^{r(U, V_0)}, \quad X^V_{const\ V_0}(U, V_0) = \frac{1}{2c_2}$$

2. $$X_{time} = \frac{1}{\sqrt{1-\frac{1}{r}}}\partial_t = \frac{1}{\sqrt{1-\frac{1}{r}}}\Big(-\frac{U}{2}\partial_U + \frac{V}{2}\partial_V\Big)$$

3. $$X^U_{sym} = X^V_{sym} = \frac{1}{2}\sqrt{r}e^{\frac{r}{2}}$$

Solutions for $\mathrm{div}(X) = 0$



(a) X^U_{con} (b) X^U_{time} (c) X^U_{sym}

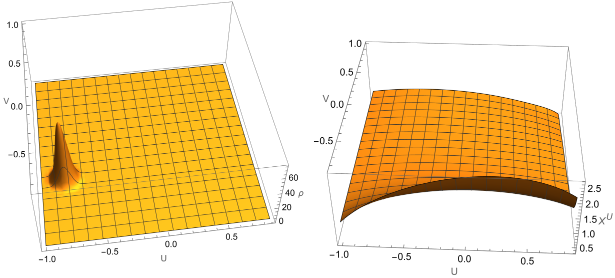
Figure: Solutions with different boundaries.

Flow through the horizon

Challenge in solving above equation to get $X(0)$:

$$\frac{r(U,V)-1}{UV} = \frac{W(-UV/e)}{UV}$$

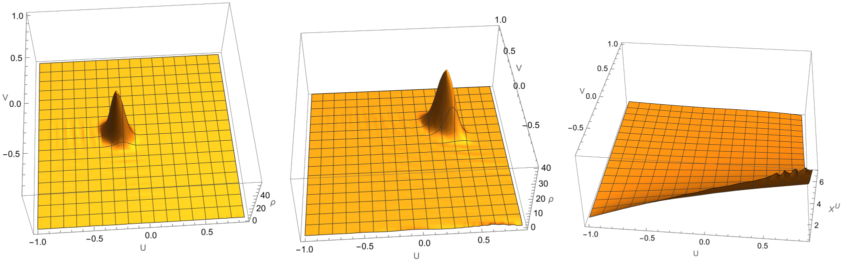
Use series expansion with accuracy within 1% for our chosen domain.
Also, used X_{sym} at the $(-4, V)$ boundary.



(a) $\rho(0)$ (b) $X^U(0)$

Figure: Initial data for horizon crossing.

Flow through the horizon



(a) $\rho(0.5)$ (b) $\rho(1)$ (c) $X^U(0.7)$

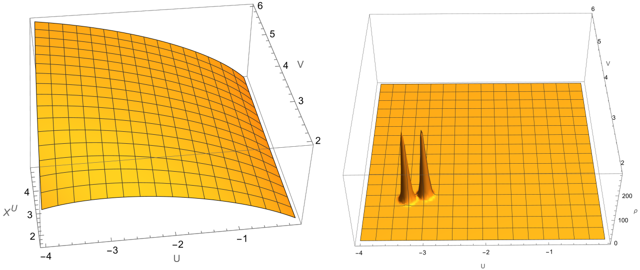
Figure: Evolved data across horizon.

Two bump flow collision

Here, we obtained $X(0)$ from above equation by using

$$3 + 2\tanh((V-2)/2)$$

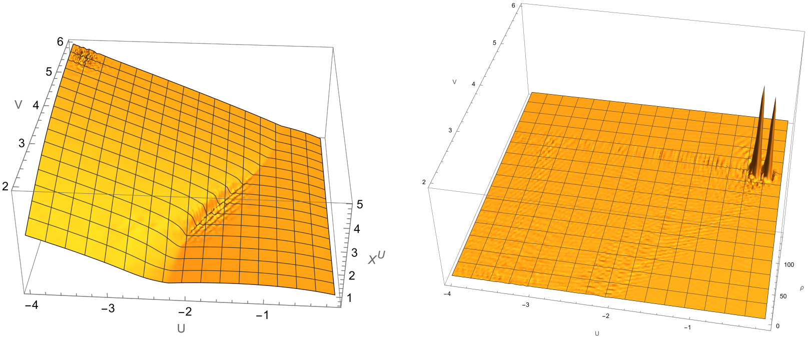
at the $U = -4$ boundary. Also, the initial bumps are close but not overlapping.



(a) $X^U(0)$ (b) $\rho(0)$

Figure: Initial data for two bumps.

Two bump flow collision

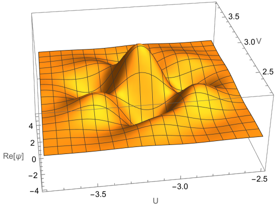


(a) $X^U(1.3)$ (b) $\rho(1.3)$

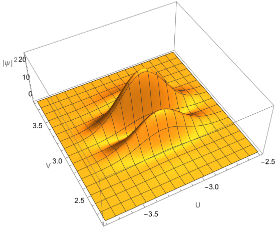
Figure: Evolved two bumps showing partial collision.

ψ interference

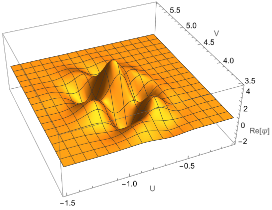
Take $\psi = e^{i(U+V)}\mathrm{bump1} + e^{i(U-V)}\mathrm{bump2}$ for interference that evolves:



(a) Re(ψ(0))



(b) $|\psi(0)|^2$



(c) Re(ψ(1))